EIC: an idea…

The 5th Berkeley school on

'Collective dynamics in high energy collisions'

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European Research Council

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Plan of the lectures

Lecture 1: Electron scattering and the structure of matter

Lecture 2: High density QCD

A more specialized document:
« Electron Ion Collider: The Next QCD Frontier »
arXiv:1212.1701

Some issues

What is the wave-function of a hadron, a nucleus, at high energy?

Relevance for heavy ion collisions: Initial versus final state (QGP) effects.

Dominance of small x gluons. Saturation effects. Are they relevant, visible?

Need for new schemes to calculate. Non linear evolution equations. CGC. How to test these? In particular transition from high pT regime to saturation regime?

Some issues

Is the saturation regime universal? Universality of the hadron cross sections at high energy?

Initial state of nucleus-nucleus collisions?
Transition to the QGP? Thermalization?

Can this be understood in terms of weak coupling ? Or are strong coupling techniques necessary ?

Etc.

LECTURE 1

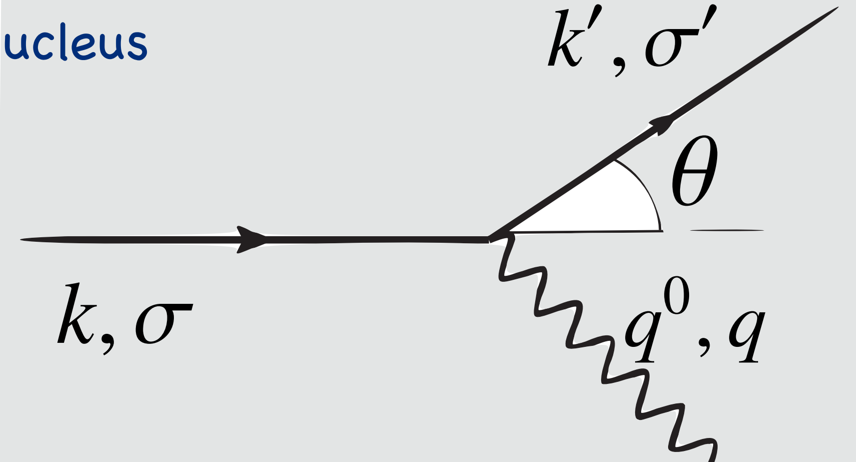
Electron scattering and the structure of matter

Why electrons ?

Electrons are 'pointlike' particles (size less than 0.001 fm)

Interaction with matter well known (QED) and weak (can be treated with perturbation theory).

In rest frame of nucleus

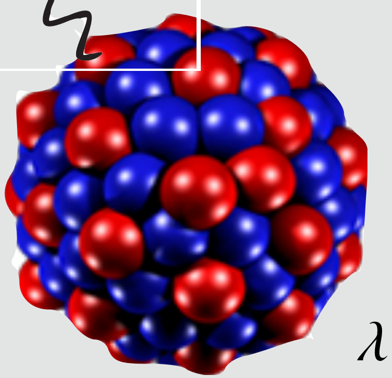


$$q = k - k'$$

$$q^0 = E - E'$$

Exchanged photon is space-like

$$-Q^2 = q_0^2 - q^2 \leq 0$$



virtuality of the exchanged photon

$$Q^2 = 4EE' \sin^2 \frac{\theta}{2}$$

$$\frac{\lambda}{2\pi} = \frac{\hbar c}{qc} = \frac{197.3}{q_{[\mathrm{MeV}]}} [\mathrm{fm}]$$

Elastic scattering

Elastic scattering on a nucleus. Ignore polarization effect (spin average). Dominated by Coulomb interaction. Non relativistic treatment.

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{el} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathbf{point}} |\langle\Psi_0|F(\boldsymbol{q})|\Psi_0\rangle|^2$$

Diffusion on a pointlike charge

Elastic form factor

Contains information on the structure of the nucleus

Mott cross section

(diffusion on a spin 1/2 point-like charged particle)

$$\sigma_{Mott} = \frac{\alpha^2 \cos^2\theta/2}{4E^2\sin^2\theta/2}$$

Elastic scattering

Elastic form factor

$$F(\boldsymbol{q}) = \sum_i \mathrm{e}^{i\boldsymbol{q}\cdot(\boldsymbol{r}_i - \boldsymbol{R})}$$

$$\langle \Psi_0 | F(\boldsymbol{q}) | \Psi_0 \rangle \approx \int \mathrm{d}^3\boldsymbol{r}\, \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{r}}\, \rho_{\mathrm{p}}(\boldsymbol{r})$$

For a pointlike particle $\quad F(\boldsymbol{q}) = 1$

For a charge distribution

$$F(\boldsymbol{q}) \approx Z \left(1 - \frac{\boldsymbol{q}^2}{6} \langle r^2 \rangle_{\mathrm{p}} + \cdots \right)$$

Note factor Z^2 in the cross section: coherence.

Some form factors

Sharp sphere

$$F(q) = 3Z\frac{\sin qR - qR\cos qR}{(qR)^3}$$

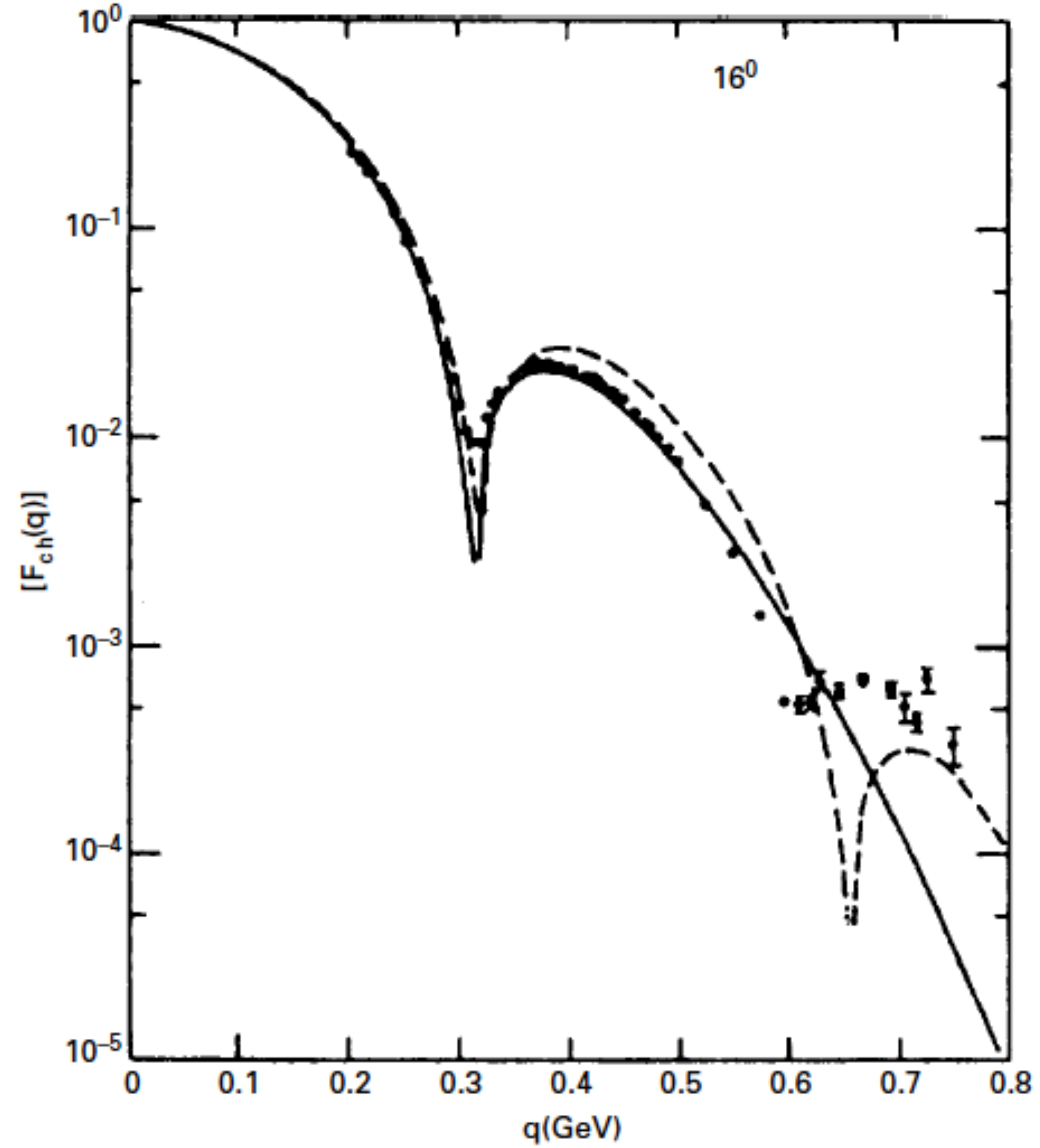
Hydrogen atom

$$F(q) = \left(\frac{1}{1+q^2a_0^2}\right)^2 \qquad \psi(r) \sim \mathrm{e}^{-r/a_0}$$

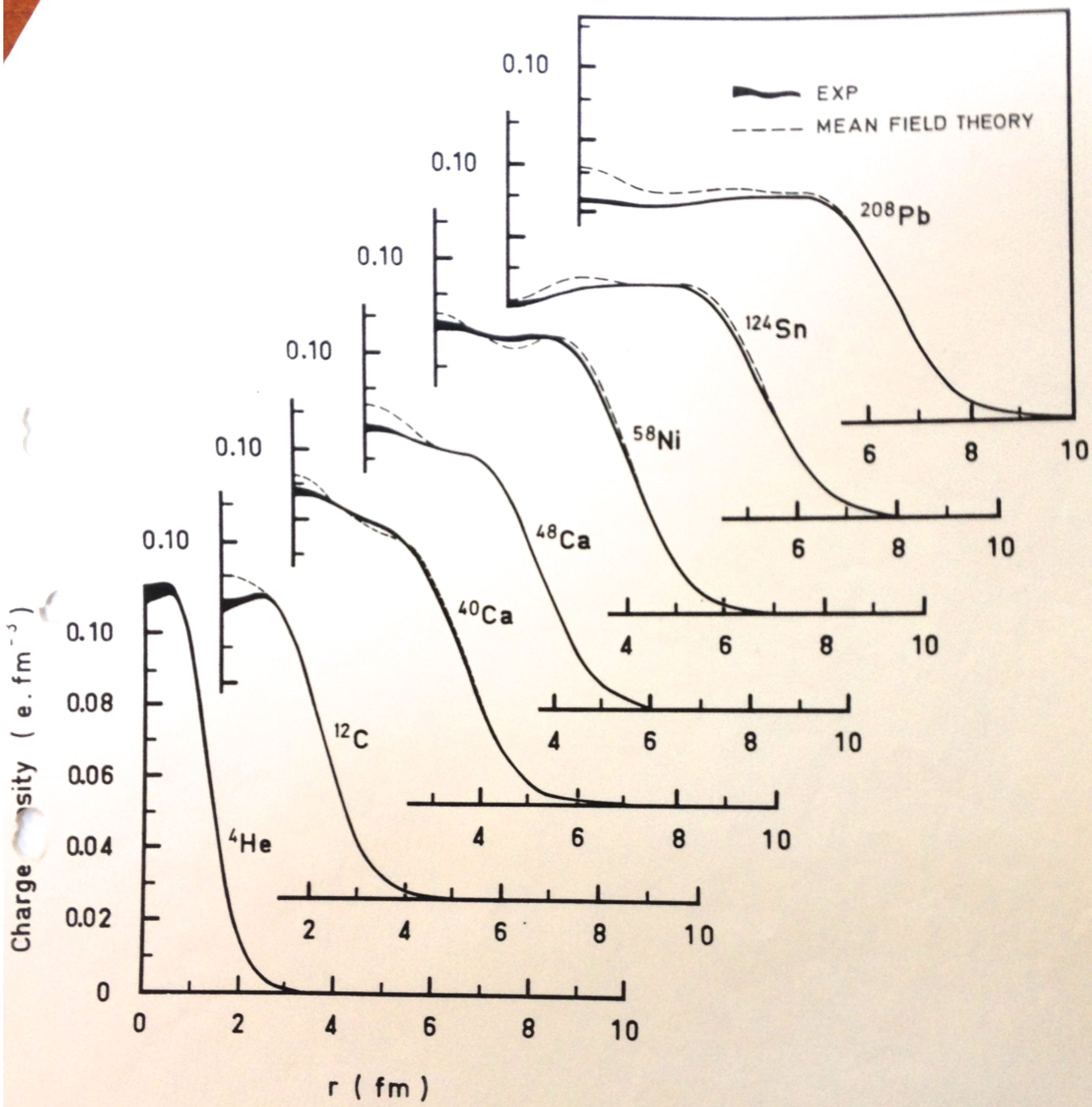
Proton

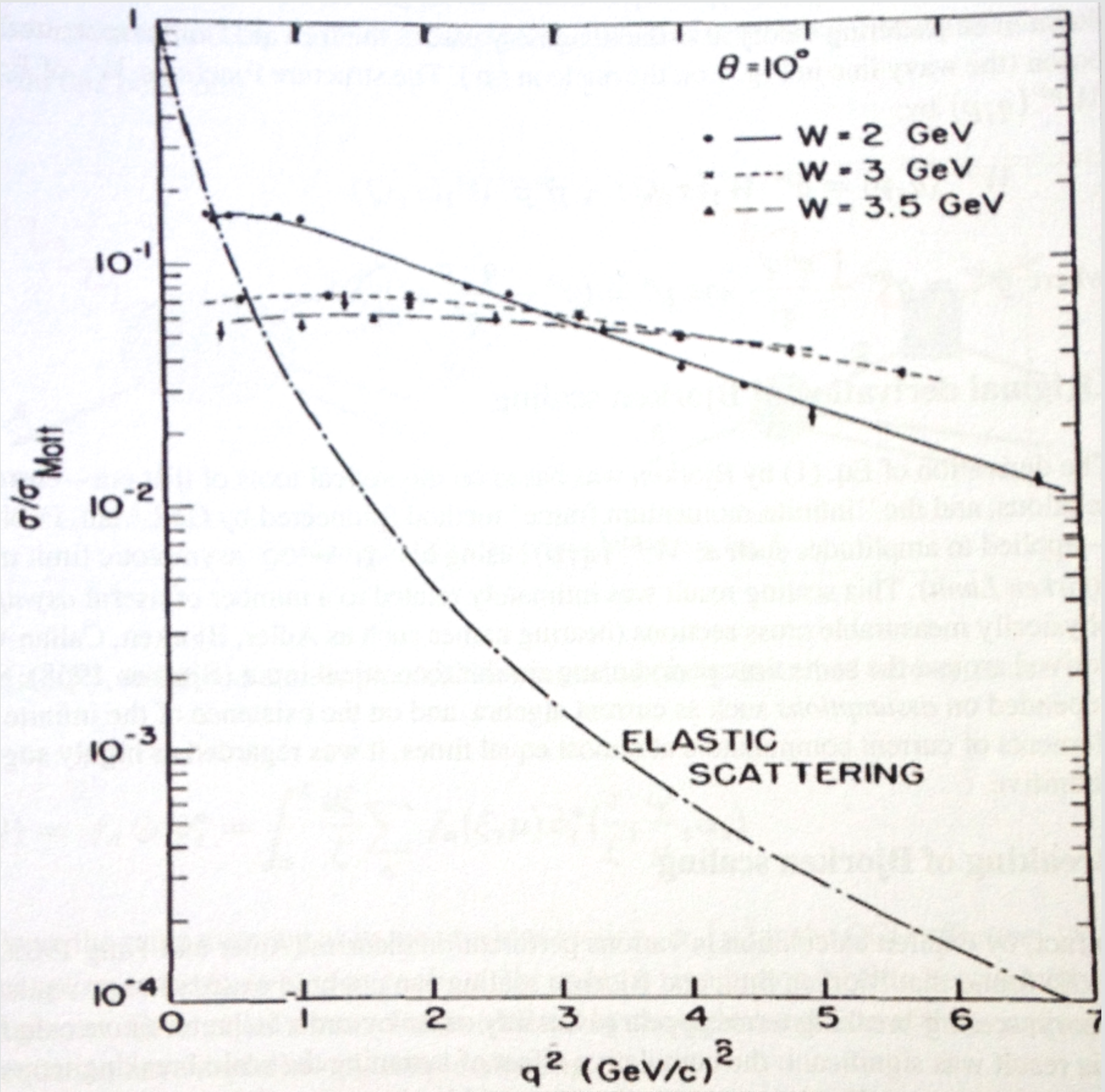
$$q_0 \approx 0.84\ \mathrm{GeV}$$

$$G_E(q^2) \approx \frac{1}{\left(1-q^2/q_0^2\right)^2} \qquad \sqrt{\langle r^2\rangle} \approx 0.83\ \mathrm{fm}$$



Elastic form factor for ^{16}O


EXP
MEAN FIELD THEORY
Charge density (e. fm⁻³)
r (fm)
⁴He
¹²C
⁴⁰Ca
⁴⁸Ca
⁵⁸Ni
¹²⁴Sn
²⁰⁸Pb
0.10
0.08
0.06
0.04
0.02
0
0 2 4 6 8 10



θ = 10°
W = 2 GeV
W = 3 GeV
W = 3.5 GeV
1
10⁻¹
10⁻²
10⁻³
10⁻⁴
σ/σ Mott
ELASTIC
SCATTERING
0
1
2
3
4
5
6
7
q² (GeV/c)²

Inelastic scattering

Same structure as elastic scattering

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{inel}} = \sigma_{\mathrm{Mott}}\, S(\boldsymbol{q},\omega) \qquad \omega = E - E'$$

Response function:

$$S(\omega,\boldsymbol{q}) = \sum_n |\langle\Psi_n|\sum_i \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{r}_i}|\Psi_0\rangle|^2\,\delta(\omega-\omega_{n0}) \qquad \omega_{n0} = E_n - E_0$$

$$S(\omega,\boldsymbol{q}) = \int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{e}^{i\omega t}\int \mathrm{d}^3\boldsymbol{r}\mathrm{e}^{-i\boldsymbol{q}\cdot\boldsymbol{r}}S(t,\boldsymbol{r})$$

$$S(t,\boldsymbol{r}) = \langle\Psi_0|\rho(t,\boldsymbol{r})\rho(0,0)|\Psi_0\rangle$$

$$\rho(\boldsymbol{q}) = \int d^3\boldsymbol{r}\,\mathrm{e}^{-i\boldsymbol{q}\cdot\boldsymbol{r}}\,\rho(\boldsymbol{r}) = \sum_i \mathrm{e}^{-i\boldsymbol{q}\cdot\boldsymbol{r}_i}$$

Sketch of a proof

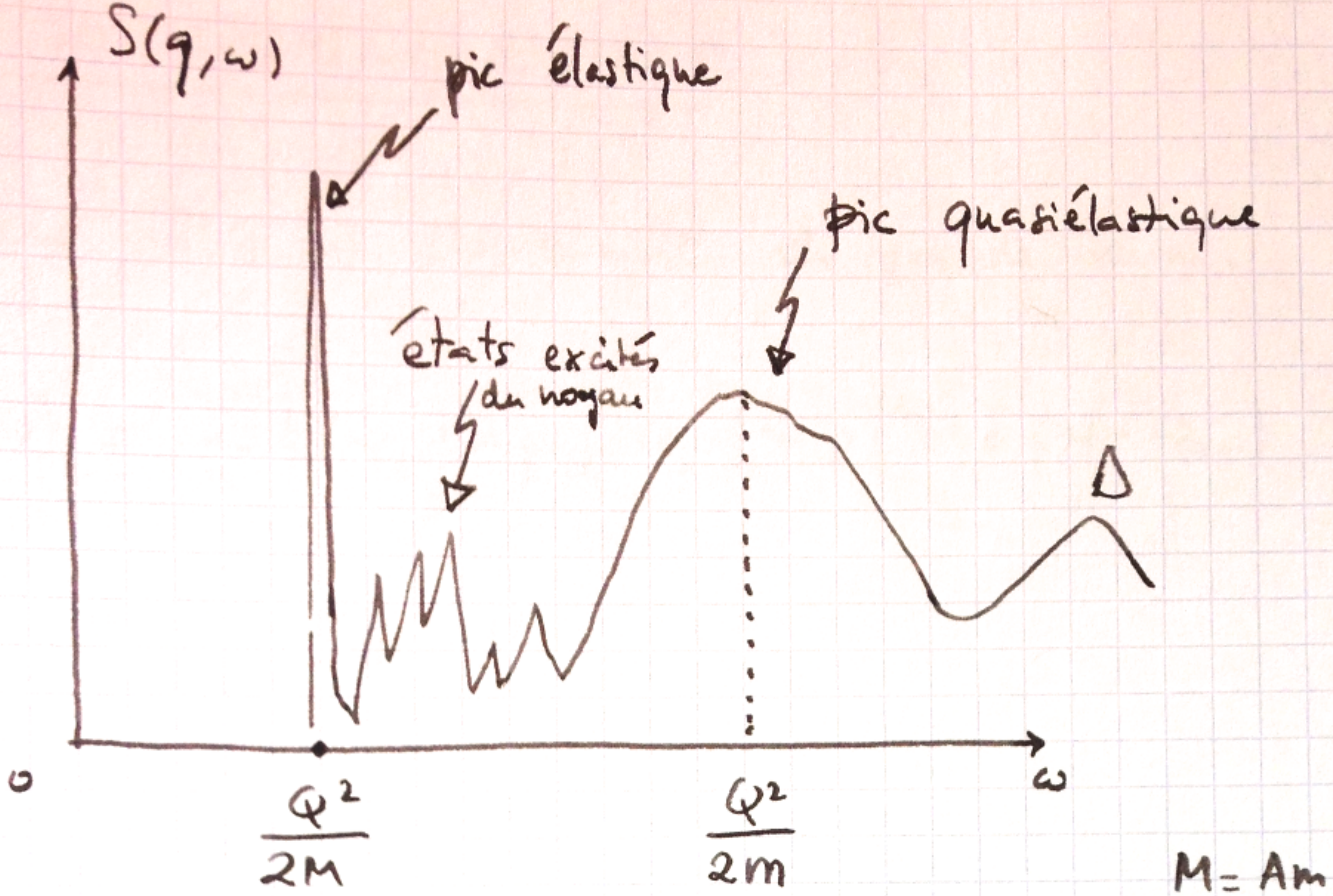
Coulomb scattering in Born approximation

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega} \propto \sum_{n,\boldsymbol{k}'} \left|\langle \Psi_n; \boldsymbol{k}'|H_{\mathrm{int}}|\psi_0; \boldsymbol{k}\rangle\right|^2 \delta(E' + E_n - E - E_0)$$

$$H_{\mathrm{int}} = \sum_i V_{\mathrm{coul}}(\boldsymbol{r} - \boldsymbol{r}_i)$$

$$\langle \Psi_n; \boldsymbol{k}'|H_{\mathrm{int}}|\psi_0; \boldsymbol{k}\rangle \propto V_{\mathrm{coul}}(\boldsymbol{q})\langle \Psi_n|\sum_i \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{r}}|\psi_0\rangle$$

$$\langle \Psi_n|\sum_i \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{r}}|\psi_0\rangle = \langle \Psi_n|\rho(-\boldsymbol{q})|\psi_0\rangle = \langle \Psi_n|J^0(-\boldsymbol{q})|\psi_0\rangle$$



S(q,ω)
pic élastique
pic quasiélastique
états excités du noyau
Δ
0
ω
Q²/2M
Q²/2m
M = Am

length and time scales

Characterizing correlations between density fluctuations

Natural length scale: nuclear size R

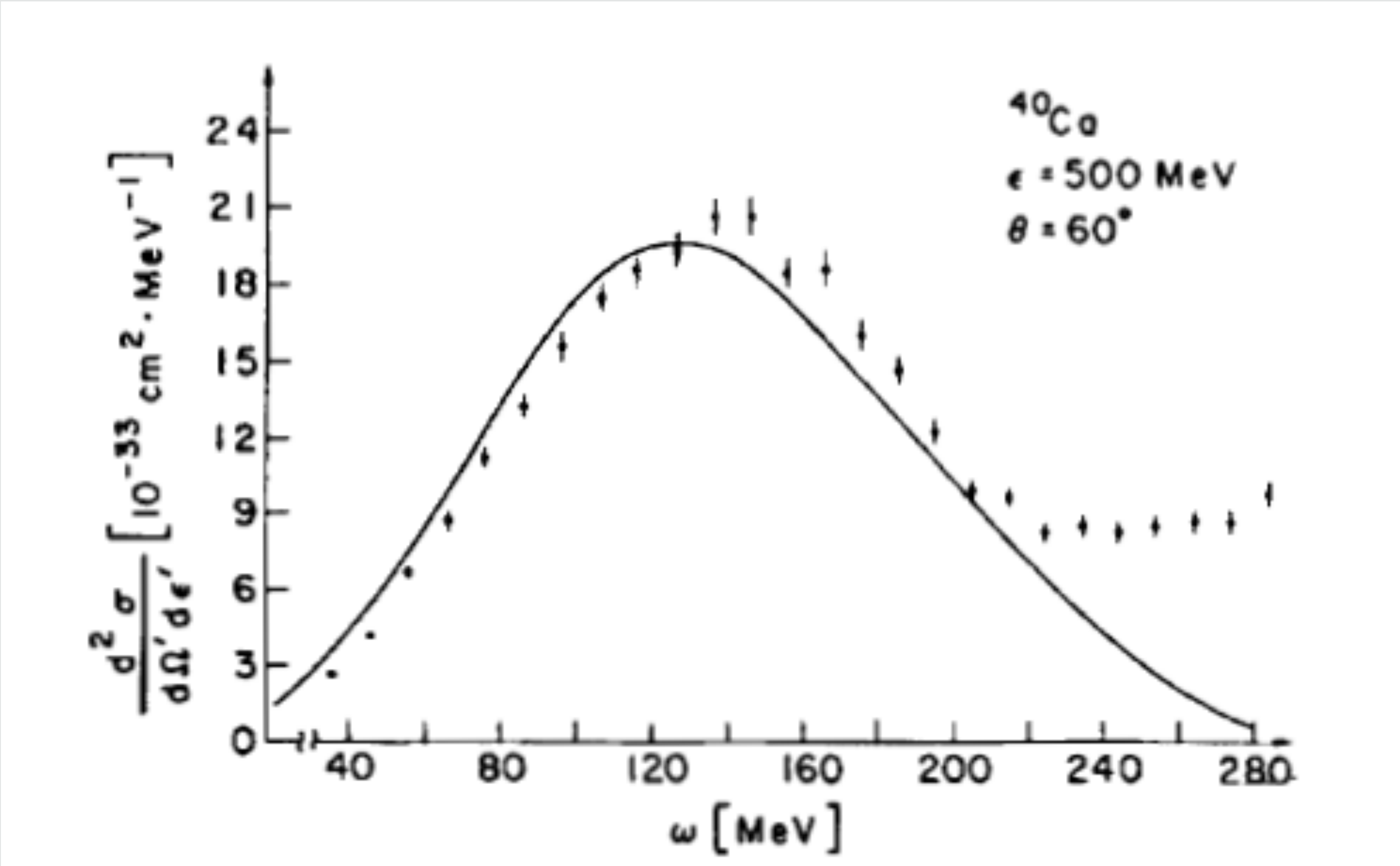
Natural time scale: $\dfrac{R}{v_F}$

If system is probed with

$$\lambda \ll R \qquad \tau \ll \frac{R}{v_F}$$

correlations are ‘invisible’: the probe scatters on constituents as if they were non interacting

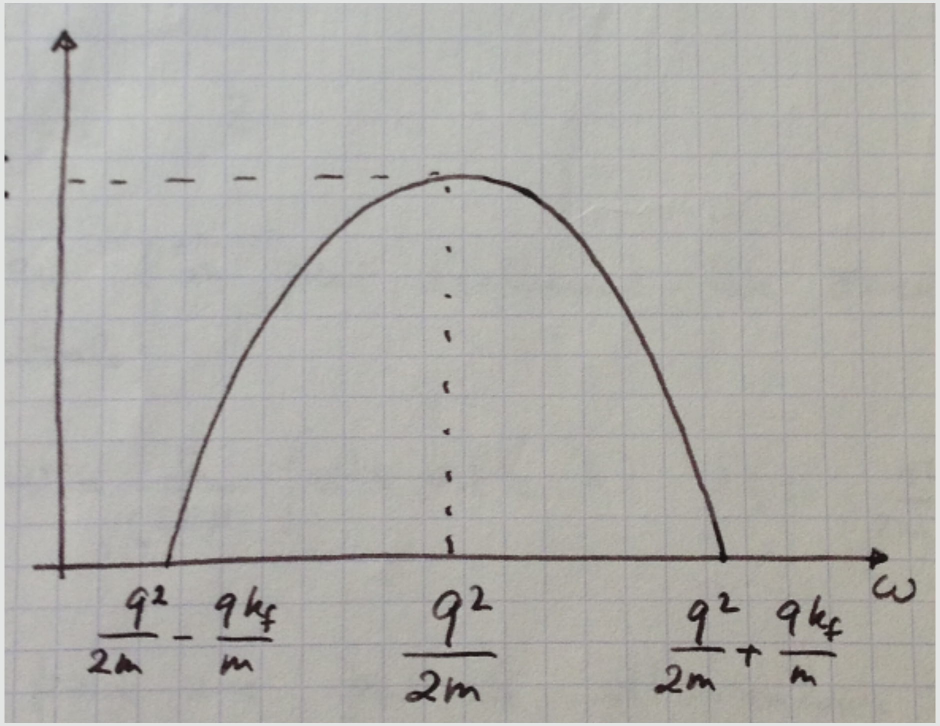
Quasi-elastic peak



Incoherent scattering on the protons of the nucleus

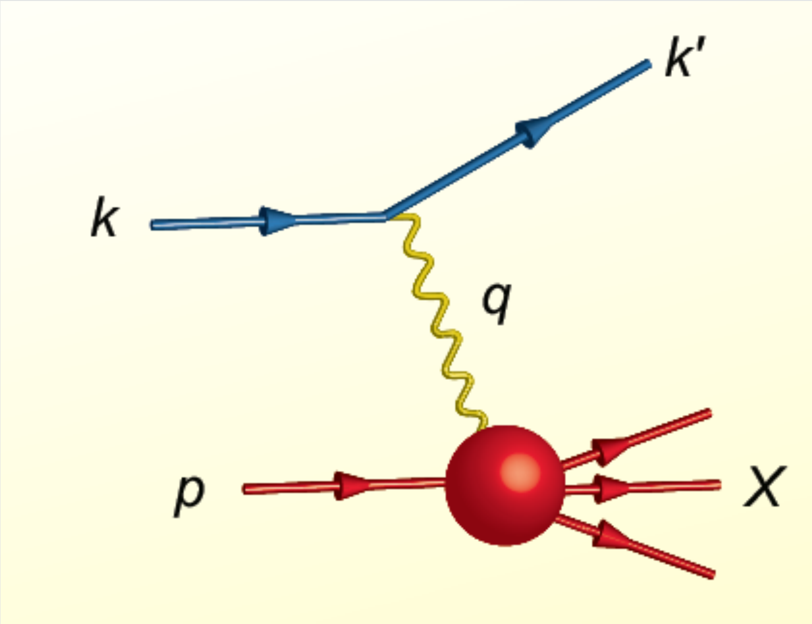
$$S(\omega, \boldsymbol{q}) = \int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^3} n(\boldsymbol{k})\, \delta\left(\omega - \frac{q^2}{2m} - \frac{\boldsymbol{q}\cdot\boldsymbol{k}}{m}\right)$$

$$S(\boldsymbol{q}, \omega) = \frac{3m}{4qk_F} Z \left[1 - \left(\frac{m\omega}{qk_F} - \frac{q}{2k_F}\right)^2\right]$$



Note the factor Z reflecting incoherence of the scattering

Deep inelastic scattering



Lorentz invariant

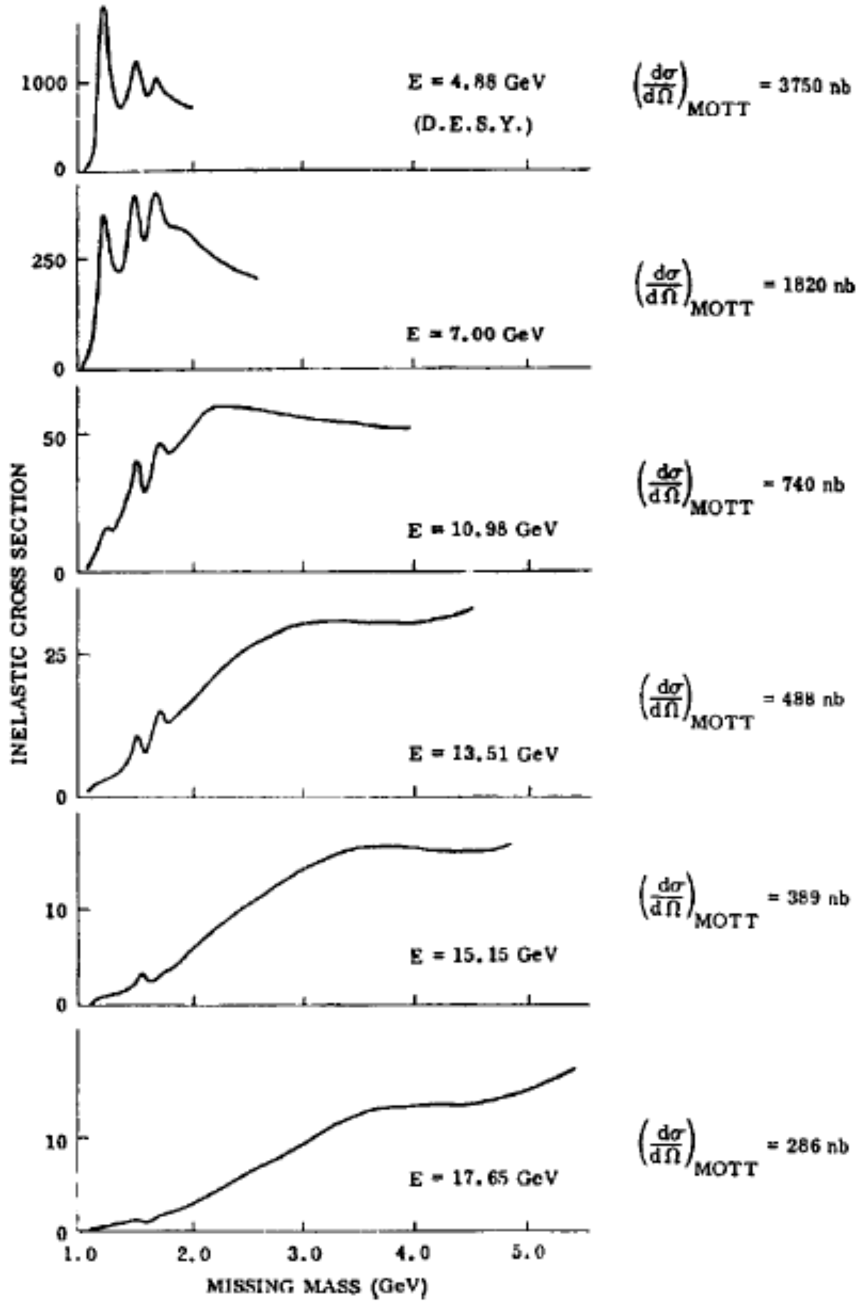
$$\nu = P \cdot q$$

$$W^2 \equiv (P+q)^2 = M^2 + 2\nu + q^2$$

In rest frame of the proton $\quad \nu = M q_0 = M(E - E')$

Bjorken x $\quad x_{\mathrm{Bj}} = \dfrac{Q^2}{2P \cdot q} \qquad 0 \le x_{\mathrm{Bj}} \le 1$

For elastic scattering $\quad x_{\mathrm{Bj}} = 1 \quad (W = M)$



INELASTIC CROSS SECTION
MISSING MASS (GeV)
E = 4.88 GeV
(D.E.S.Y.)
E = 7.00 GeV
E = 10.98 GeV
E = 13.51 GeV
E = 15.15 GeV
E = 17.65 GeV
$\left(\frac{d\sigma}{d\Omega}\right)_{MOTT}$ = 3750 nb
$\left(\frac{d\sigma}{d\Omega}\right)_{MOTT}$ = 1820 nb
$\left(\frac{d\sigma}{d\Omega}\right)_{MOTT}$ = 740 nb
$\left(\frac{d\sigma}{d\Omega}\right)_{MOTT}$ = 488 nb
$\left(\frac{d\sigma}{d\Omega}\right)_{MOTT}$ = 389 nb
$\left(\frac{d\sigma}{d\Omega}\right)_{MOTT}$ = 286 nb

The inclusive cross section takes the form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E'\mathrm{d}\Omega} = \alpha^2 \frac{\cos^2\frac{\theta}{2}}{4E^2\sin^4\frac{\theta}{2}} \left(W_2(Q^2,\nu) + 2\tan^2\frac{\theta}{2} W_1(Q^2,\nu) \right)$$

Generalization of the response function

$$W^{\mu\nu} = \frac{1}{4\pi m} \int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot x} \langle P | J^\mu(x) J^\nu(0) | P \rangle$$

$$W^{\mu\nu} = -W_1 \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) + \frac{W_2}{m^2} \left(P^\mu - \frac{P\cdot q}{q^2} q^\mu \right) \left(P^\mu - \frac{P\cdot q}{q^2} q^\nu \right)$$

Incoherent scattering on point-like (spin 1/2) constituents

$$W_2 = \delta\left(\nu - \frac{Q^2}{2M}\right) \qquad\qquad 2W_1 = \frac{Q^2}{2m^2}\delta\left(\nu - \frac{Q^2}{2m}\right)$$

The delta functions reflect energy momentum conservation

One can write

$$\nu W_2(Q^2,\nu) = \delta\left(\frac{Q^2}{2m\nu} - 1\right) = \delta(x_{\mathrm{Bj}} - 1)$$

$$2mW_1(Q^2,\nu) = \frac{Q^2}{2m\nu}\delta\left(\frac{Q^2}{2m\nu} - 1\right) = x_{\mathrm{Bj}}\delta(x_{\mathrm{Bj}} - 1)$$

Scaling. No scale !

Length and time scales (again)

Infinite momentum frame (for the proton)

$$P = (P, 0_\perp, P)$$

Typical time scale characterizing the parton motion

$$\tau_{\text{partons}} \sim \frac{1}{k_\perp}\frac{P}{m} \sim \frac{P}{\Lambda^2_{\text{QCD}}}$$

typical parton time scales are Lorentz dilated

Choose (Breit frame) $\quad q^\mu = (q^0, q_\perp, 0)$

$$P \cdot q = Pq^0 = \frac{Q^2}{2x_{\text{Bj}}} \qquad q^0 = \frac{Q^2}{2x_{\text{Bj}}P}$$

Duration of DIS process $\quad \tau_{\text{DIS}} \sim \frac{1}{q^0} \approx \frac{2x_{\text{Bj}}P}{Q^2} \ll \tau_{\text{partons}} \qquad (Q^2 \gg \Lambda^2_{\text{QCD}})$

Besides $\quad Q^2 = -q_0^2 + q_\perp^2 \approx q_\perp^2$

so that the virtual photon probes transverse sizes $\Delta x_\perp \sim 1/Q$

Pre-QCD parton model

The proton is a collection of point-like fermions,

A parton of type i, carrying a fraction xF of the total proton momentum contributes

$$4\pi W_i^{\mu\nu} = 2\pi x_F \delta(x_F - x)\, e_i^2 \times \left[-\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) + \frac{2x_F}{P \cdot q} \left(P^\mu - q^\mu \frac{P \cdot q}{q^2} \right) \left(P^\nu - q^\nu \frac{P \cdot q}{q^2} \right) \right]$$

If there are $f_i(x_F)dx_F$ partons of type i with a momentum fraction between x_F and $x_F + dx_F$, we have

$$W^{\mu\nu} = \sum_i \int_0^1 \frac{dx_F}{x_F} f_i(x_F)\, W_i^{\mu\nu} , \quad F_1 = \frac{1}{2} \sum_i e_i^2 f_i(x) , \quad F_2 = 2xF_1$$

What about QCD

The parton picture emerges without specific reference to the actual dynamics.

Asymptotic freedom does not seem required...

Dynamics enter in specific deviations from the simple parton model (scaling violations)

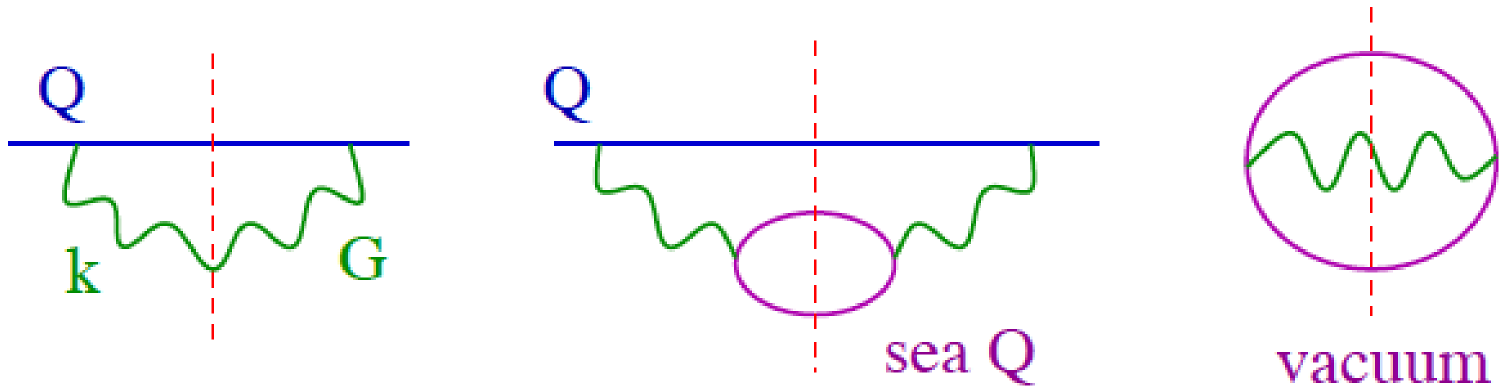
The « wave function » of the proton (or nucleus) depends on frame, depends on probe, etc.

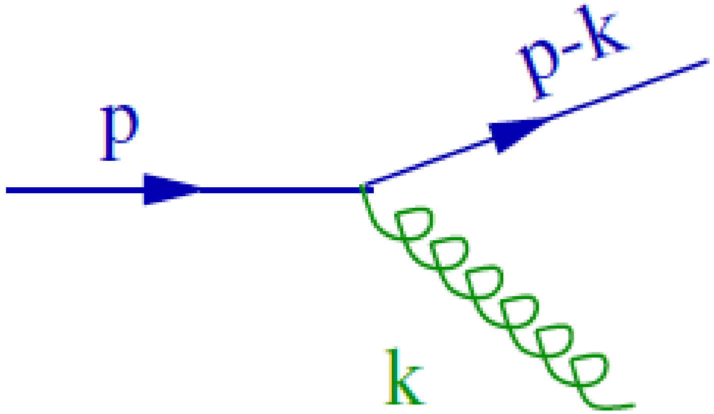
Relativity is important

Constituents: nucleons, valence quarks, gluons, sea quarks and antiquarks

Light cone wave function

$$|\mathrm{proton}> \;=\; c_0|QQQ> + c_1|QQQG> + c_2|QQQ\,sQ\,sQ> + \cdots + c_{vac}|QQQ\mathrm{vacuum}>$$

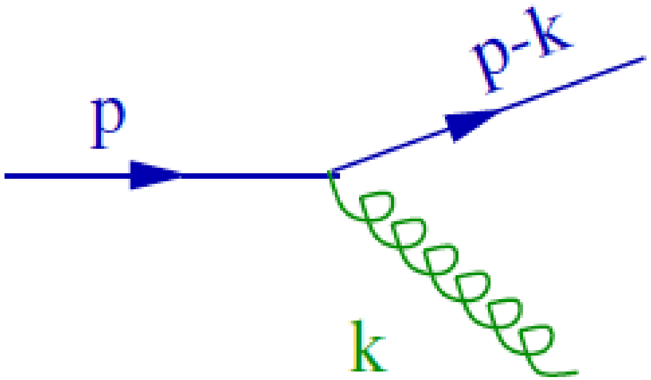




$$\mathrm{d}\mathcal{P} \simeq \alpha_s \frac{\mathrm{d}k_\perp^2}{k_\perp^2} \frac{\mathrm{d}x}{x}$$

One can calculate the change of the wave-function, not the wf itself. One needs « initial conditions »

Radiation and multiplication of partons



$$\mathrm{d}\mathcal{P} \simeq \alpha_s \frac{\mathrm{d}k_\perp^2}{k_\perp^2} \frac{\mathrm{d}x}{x}$$

E.g., for a single valence quark (perturbation theory)

$$xG(x, Q^2) = \frac{\alpha C_F}{\pi} \ln\left(\frac{Q^2}{\mu^2}\right)$$

Gluon density

$$\frac{xG(x, Q^2)}{\pi R^2} = \int^{Q^2} d^2\mathbf{k}_\perp \frac{dN}{dy d^2\mathbf{k}_\perp d^2\mathbf{b}}$$

$$f(x, \mathbf{k}_\perp, \mathbf{b}) \equiv \frac{(2\pi)^3}{2(N_c^2 - 1)} \frac{dN}{dy d^2\mathbf{k}_\perp d^2\mathbf{b}}$$

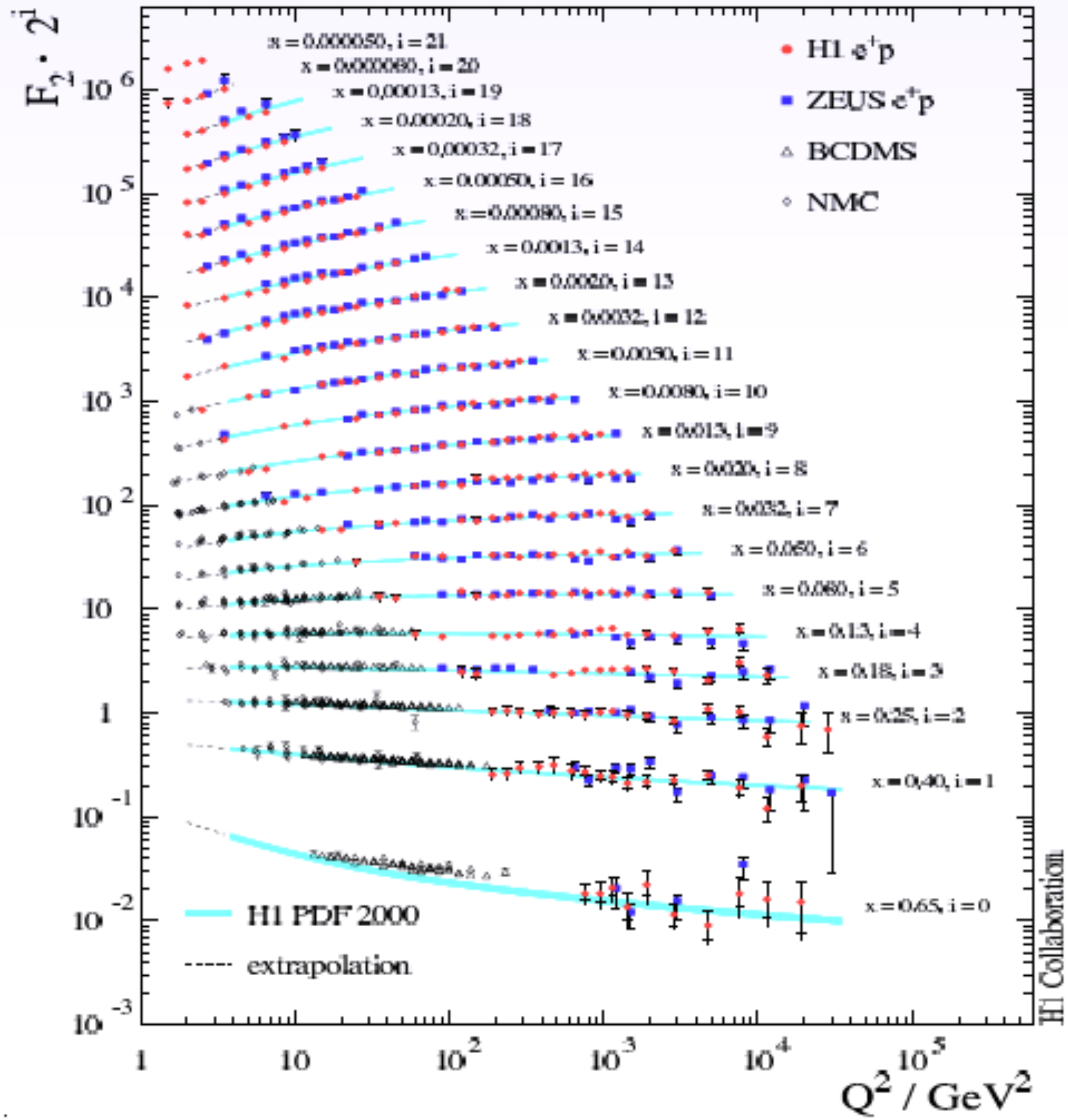
What QCD tells us

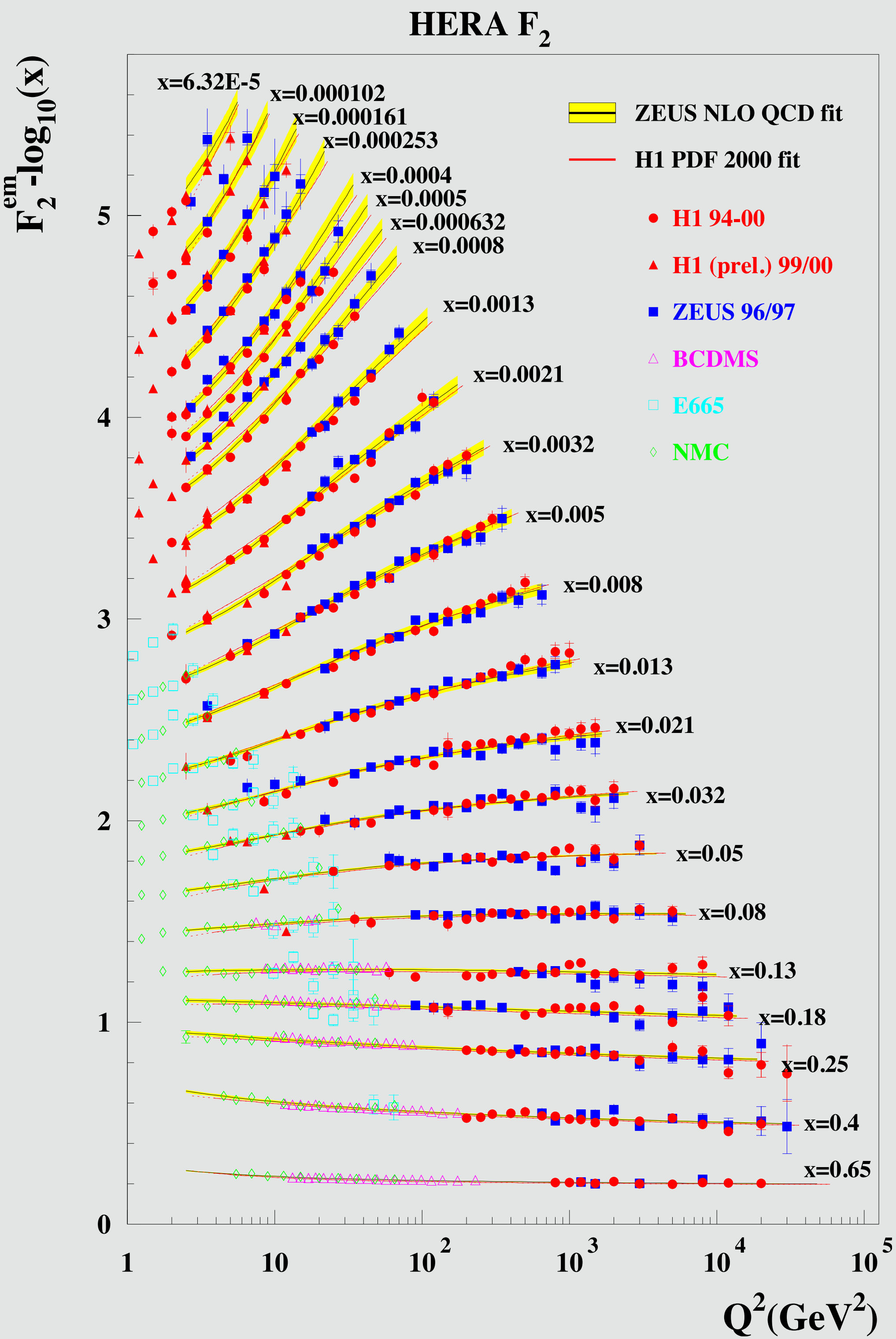
Asymptotic freedom leads to specific violations of the naive parton model: Q^2 dependence of the structure functions.

The parton distributions are non perturbative, but their dependence on x and Q^2 can be calculated with perturbation theory (from non perturbative initial conditions). Evolution equations (DGLAP, BFKL, etc).

The parton distributions are universal, i.e., they are the same in all inclusive processes.

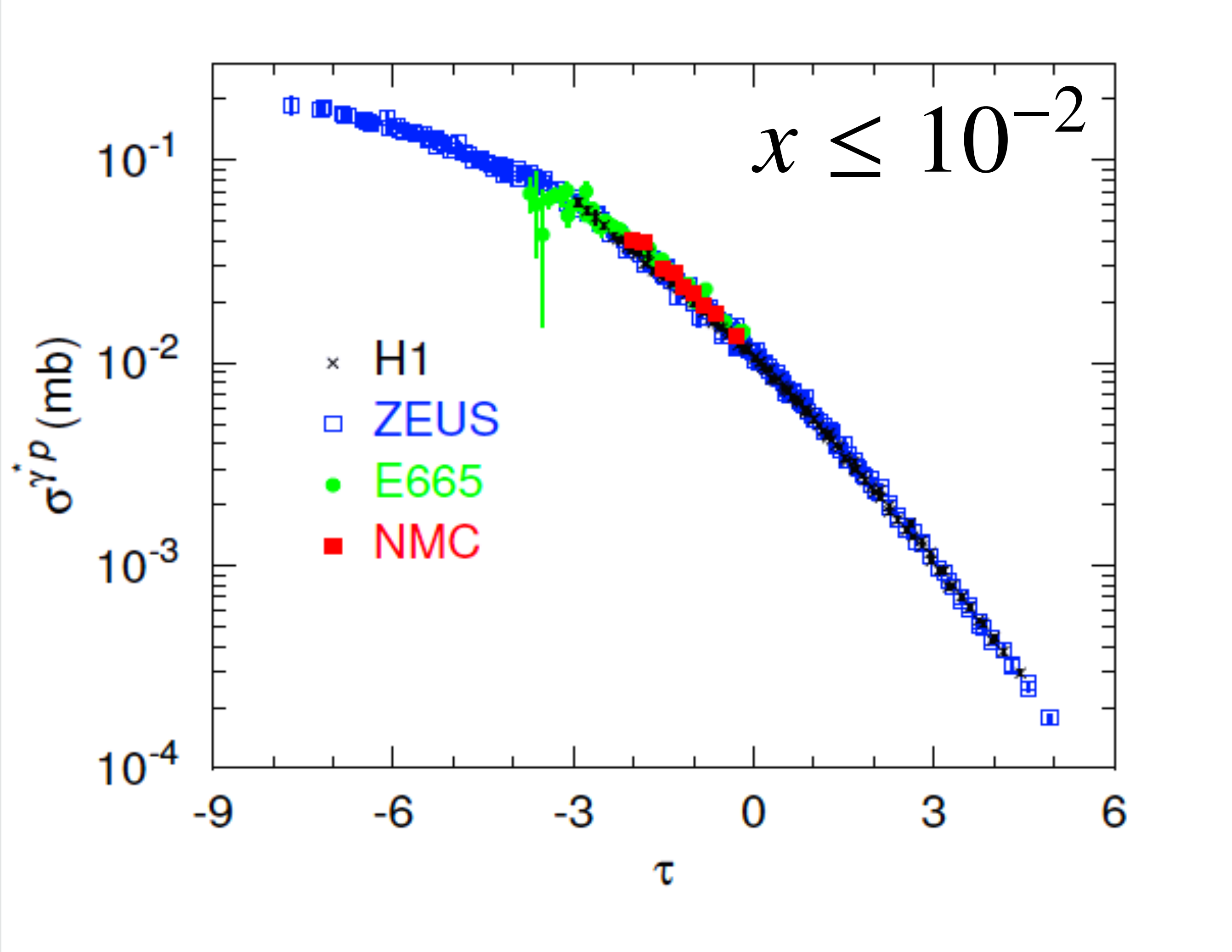
DIS results for F_2 and DGLAP fit at NLO :





HERA F2
F2em-log10(x)
x=6.32E-5
x=0.000102
x=0.000161
x=0.000253
x=0.0004
x=0.0005
x=0.000632
x=0.0008
x=0.0013
x=0.0021
x=0.0032
x=0.005
x=0.008
x=0.013
x=0.021
x=0.032
x=0.05
x=0.08
x=0.13
x=0.18
x=0.25
x=0.4
x=0.65
ZEUS NLO QCD fit
H1 PDF 2000 fit
H1 94-00
H1 (prel.) 99/00
ZEUS 96/97
BCDMS
E665
NMC
Q2(GeV2)

Same data plotted differently



$$\tau = \log(x^{0.32} Q^2)$$

Length and time scales (again)

We want to characterize the typical time and longitudinal distances involved in

$$\int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot x} \langle P|J^\mu(x) J^\nu(0)|P\rangle$$

Light cone coordinates

$$x^\pm = \frac{x^0 \pm x^3}{\sqrt{2}} \qquad q^\pm = \frac{q^0 \pm q^3}{\sqrt{2}} \qquad q^0 x^0 - q^3 x^3 = q^+ x^- + q^- x^+$$

DIS in proton rest frame, with $q^\mu = (q^0, 0_\perp, q^3)$

We have $\nu = P \cdot q = M q^0$ so that $q^0 = \dfrac{Q^2}{2M x_{\mathrm{Bj}}} \gg Q$

Also, $0 \leq Q^2 = (q^3)^2 - (q^0)^2$ hence $q^0 \approx q^3 \gg Q$

Therefore

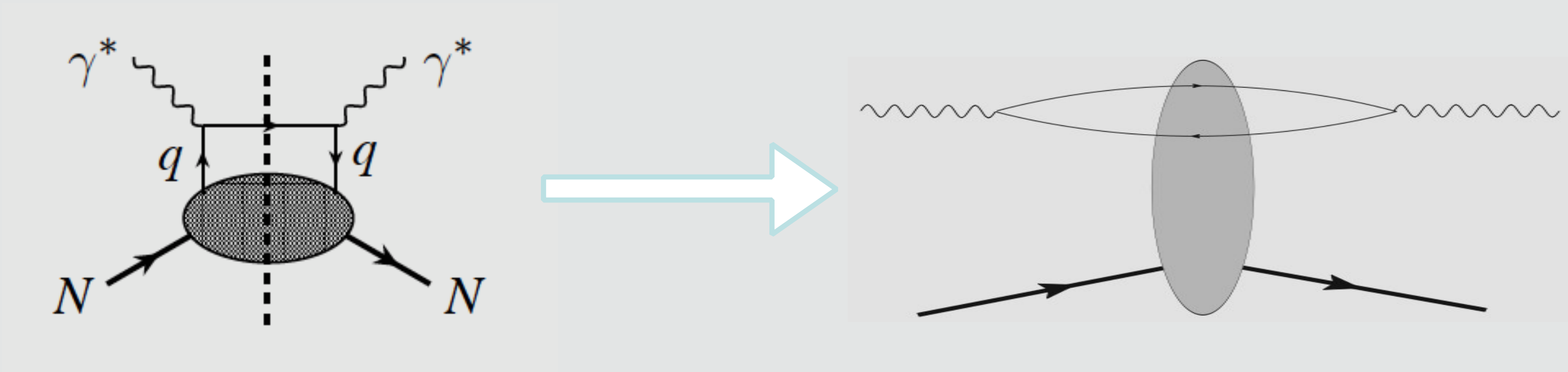
$$q^+ = \frac{q^0 + q^3}{\sqrt{2}} \approx \sqrt{2} q^0 \qquad q^- = \frac{q^0 - q^3}{\sqrt{2}} = \frac{q^+ q^-}{q^+} \approx -\frac{Q^2}{2\sqrt{2} q^0} = -\frac{M x_{\rm Bj}}{\sqrt{2}}$$

It follows that

$$x^+ \sim \frac{1}{|q^-|} \approx \frac{\sqrt{2}}{M x_{\rm Bj}} \gtrsim \frac{1}{\Lambda_{\rm QCD}} \qquad x^- \sim \frac{1}{q^+} \approx \frac{M x_{\rm Bj}}{\sqrt{2} Q^2} \ll \frac{1}{\Lambda_{\rm QCD}}$$

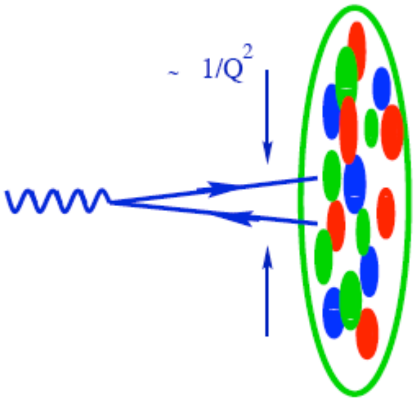
that is, $t \approx z$ and $x^+ = \frac{t + z}{\sqrt{2}} \approx \sqrt{2} t \approx \frac{\sqrt{2}}{M x_{\rm Bj}}$

If $x_{\rm Bj} \ll 1$ $t_{\rm DIS} \sim 1/(M x_{\rm Bj})$ can be much greater than $1/\Lambda_{\rm QCD}$



Geometrical scaling

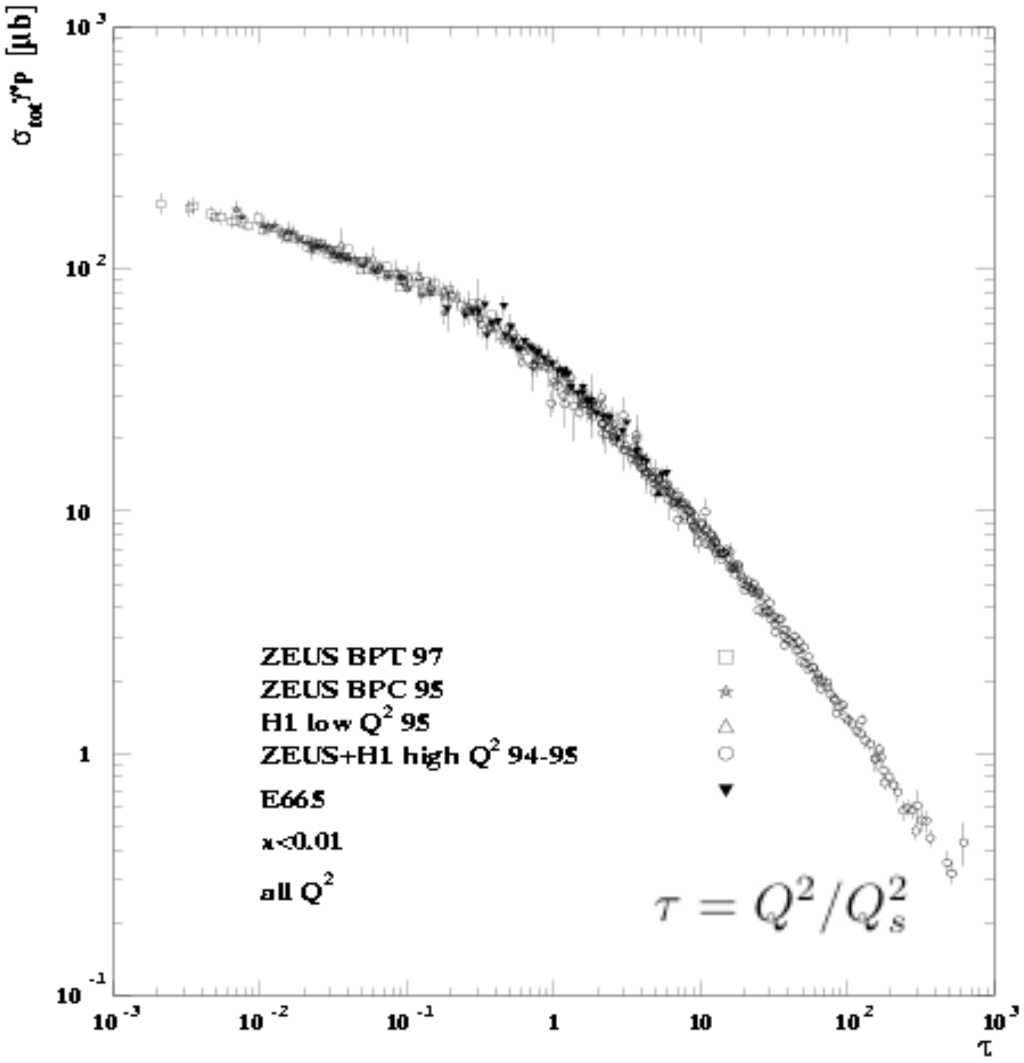
$$\sigma_{\gamma^* A}(x,Q^2) = \int dz \int d^2\boldsymbol{r}\ \underbrace{|\psi(z,\boldsymbol{r};Q^2)|^2}_{QED}\ \underbrace{2\int d^2\boldsymbol{b}\ (1 - S_\tau(\boldsymbol{x},\boldsymbol{y}))}_{\sigma_{dipole}}$$



$$\sigma_{dip} = \sigma_0\left[1 - \exp(-r_\perp^2 Q_s(x)^2)\right]$$

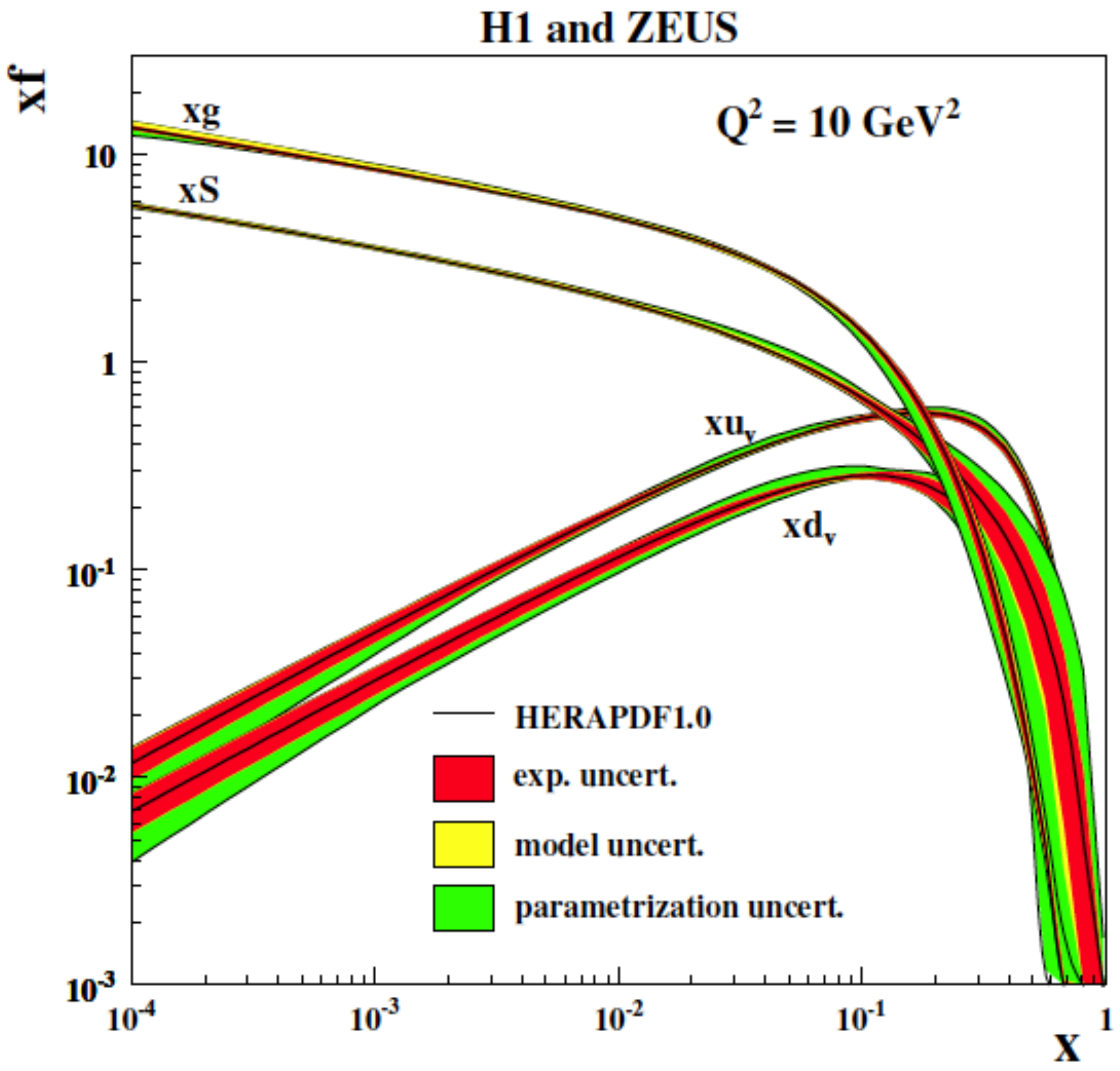
$$Q_s^2(x) \equiv Q_0^2\left(\frac{x_0}{x}\right)^\lambda$$

(Golec-Biernat,Kwiecinski,Stasto)



Lecture 2: High density QCD

Gluon density is large at small x



Non linear effects in QCD

Occur e.g. in

- Physics of the quark gluon plasma
- High density of small x gluons

A system can be strongly coupled (or strongly interacting) even when the coupling constant is small

Non linear effects in QCD when (typically)

$$\partial_\mu \sim g A_\mu$$

or equivalently

$$\langle (\partial \cdot A)^2 \rangle \sim g^2 \langle A^2 \rangle^2$$

Quantum ChromoDynamics

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \sum_f \bar{\psi}_f(i\not{D} - m_f)\psi_f$$

$$D_\mu = \partial_\mu - igA_\mu$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$$

Quark-gluon plasma

$$\langle(\partial\cdot A)^2\rangle \sim g^2\langle A^2\rangle^2$$

Thermal fluctuations

$$\langle A^2\rangle_\kappa \sim \int^\kappa \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{n_k}{k} \sim T\kappa \qquad n_k = \frac{1}{\mathrm{e}^{k/T}-1} \sim \frac{T}{k} \quad (k \lesssim T)$$

One can define an expansion parameter

$$\gamma_\kappa = \frac{g^2\langle A^2\rangle}{\kappa^2} \qquad \gamma_\kappa \sim \frac{g^2T}{\kappa}$$

For short wavelength modes $\gamma_T \sim g^2$ and one can use perturbation theory

However, for $\kappa \sim g^2T$ we have $\gamma_\kappa \sim \mathcal{O}(1)$

Long wavelength modes are strongly coupled, and highly occupied $n_\kappa \sim 1/g^2$ whatever the strength of the coupling

Weakly AND strongly coupled ...

The QGP is a multiscale system

Degrees of freedom with different wavelengths are differently coupled.

Expansion parameter depends on magnitude of thermal fluctuations and on their wavelengths

$$\gamma_\kappa = \frac{g^2 \langle A^2 \rangle_\kappa}{\kappa^2}$$

Gluon saturation

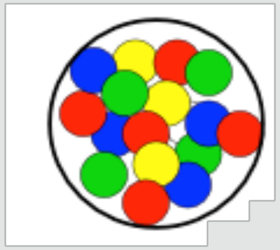
Gluon phase space density $$n_g = \frac{(2\pi)^3}{2(N_c^2-1)} \frac{\mathrm{d}N_g}{\mathrm{d}^3k\,\mathrm{d}^3b}$$

Small x gluons

$$xG(x,Q^2) = x\frac{\mathrm{d}N_g}{\mathrm{d}x} \approx \frac{\mathrm{d}N_g}{\mathrm{d}k_z\mathrm{d}b_z} \qquad \mathrm{d}k_z\mathrm{d}b_z \approx \frac{\mathrm{d}k_z}{k_z} = \frac{\mathrm{d}x}{x}$$

Recall typical transverse size of small x gluons $$\Delta x_\perp \sim \frac{1}{Q}$$

$$\frac{xG(x,Q^2)}{\pi R^2} = \int^{Q^2} \frac{d^2k_\perp}{(2\pi)^3} n_g = \langle A^2 \rangle_Q$$



Non linear effects expected when

$$g^2\langle A^2\rangle_Q \sim Q^2$$

Saturation momentum

$$Q_s^2 \approx \alpha_s \frac{xG(x,Q_s^2)}{\pi R^2}$$

gluon saturation for

$$k_\perp \le Q_S \qquad n_g \sim \frac{1}{\alpha_s}$$

The saturation scale Q_s

$$Q_s^2 \sim \alpha_s(Q_s^2)\frac{xG(x,Q_s^2)}{\pi R^2}$$

From fit to DIS (HERA)

$$Q_s^2(x) \equiv Q_0^2\left(\frac{x_0}{x}\right)^{\lambda}$$

$Q_0 = 1\mathrm{GeV}$ $\quad x_0 = 3\times 10^{-4}$

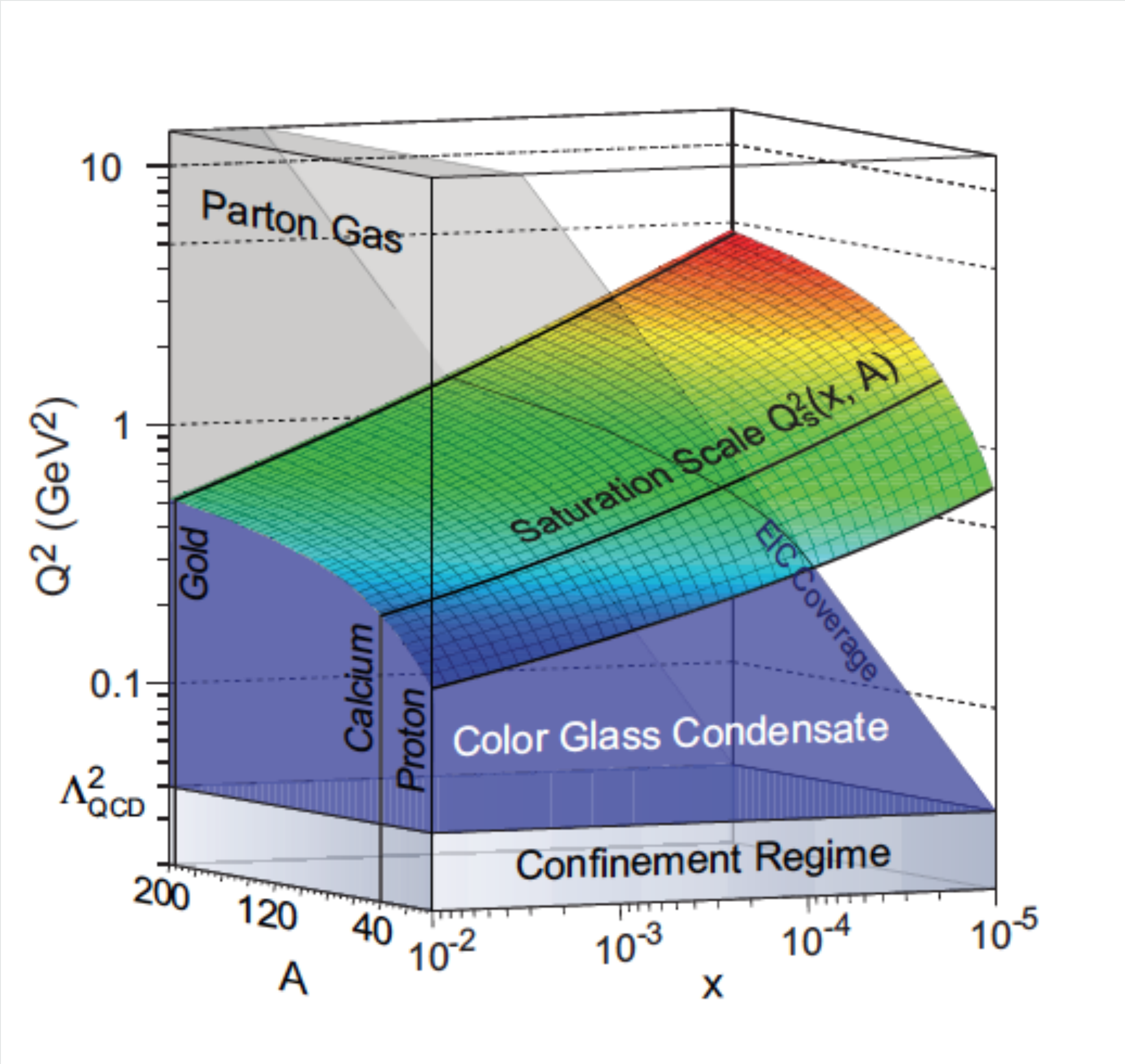
$\lambda \approx 0.3$

In a nucleus

$$\frac{xG_A(x,Q^2)}{\pi R^2} \sim A^{1/3}$$

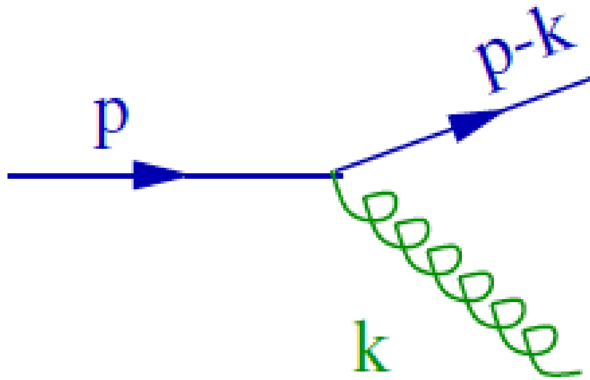
$Q_0^2 \to Q_0^2 A^{1/3}$ $\quad x = 10^{-2} \to Q_s^2 \approx 2\mathrm{GeV}^2$ $\quad (\mathrm{for}\, A = 200)$

$$Q_s^2(x) \equiv Q_0^2 \left(\frac{x_0}{x}\right)^{\lambda}$$



'Wave-function' of a nucleus
at very high energy
Linear evolution equations
and onset of saturation

Radiation and multiplication of partons



$$d\mathcal{P} \simeq \alpha_s \frac{dk_\perp^2}{k_\perp^2} \frac{dx}{x}$$

E.g., for a single valence quark (perturbation theory)

$$xG(x, Q^2) = \frac{\alpha C_F}{\pi} \ln\left(\frac{Q^2}{\mu^2}\right)$$

When $\alpha_s \ln Q^2 \sim 1$ leading order perturbation theory is not enough. Resummation is needed

$\longrightarrow$ DGLAP cascade

Evolution equations

$$\alpha_s \ln Q^2 \sim 1 \qquad \text{(DGLAP)}$$

$$Q^2 \frac{\partial}{\partial Q^2} G(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} P(x/z) G(z, Q^2)$$

$$\alpha_s \ln Q^2 \ln(1/x) \sim 1 \qquad \text{(DLL)}$$

$$\frac{\partial^2 \, xG(x, Q^2)}{\partial \ln(1/x) \partial Q^2} = \frac{\alpha C_A}{\pi} xG(x, Q^2)$$

$$xG(x, Q^2) \propto \exp\left\{2\sqrt{\bar{\alpha} \ln \frac{1}{x} \ln \frac{Q^2}{Q_0^2}}\right\}$$

$\alpha_s \ln(1/x) \sim 1$ (BFKL)

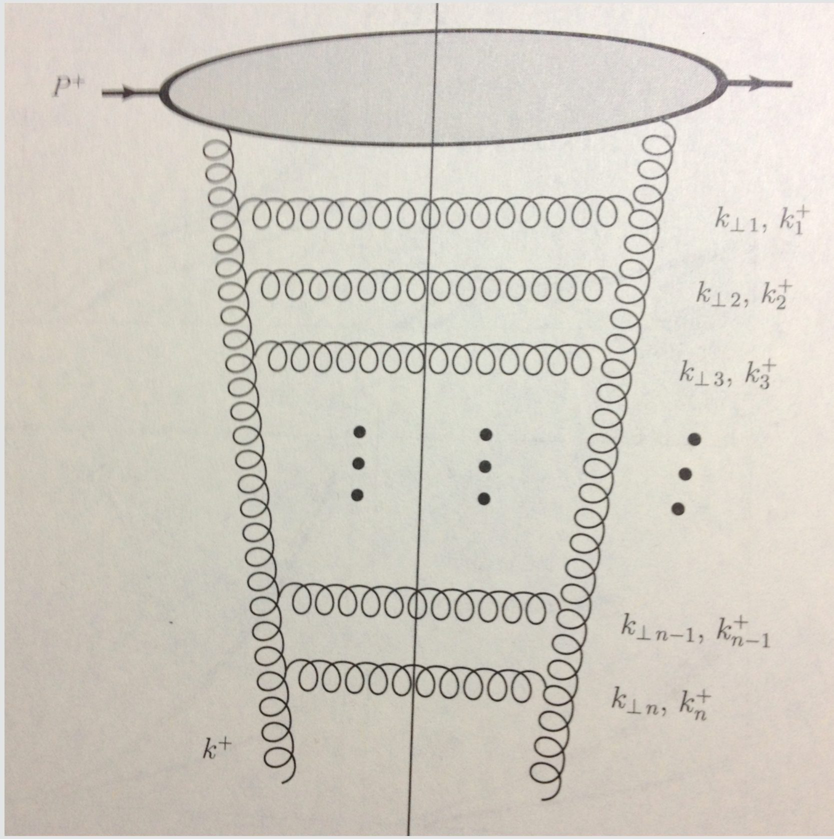
$$\frac{\partial f(y,\mathbf{k}^2)}{\partial y} = \frac{\alpha_s N_c}{\pi} \int \frac{d^2\mathbf{p}}{\pi} \frac{\mathbf{k}^2}{\mathbf{p}^2(\mathbf{k}-\mathbf{p})^2} \left[f(y,\mathbf{p}^2) - \frac{1}{2} f(y,\mathbf{k}^2) \right]$$

Exponential growth of density at small x

$$f(y,\mathbf{k}_\perp^2) \sim \mathrm{e}^{\omega\bar{\alpha}y} \sim \left(\frac{1}{x}\right)^{\omega\bar{\alpha}} \qquad \left(\bar{\alpha} \equiv \frac{\alpha_s N_c}{\pi}\right)$$

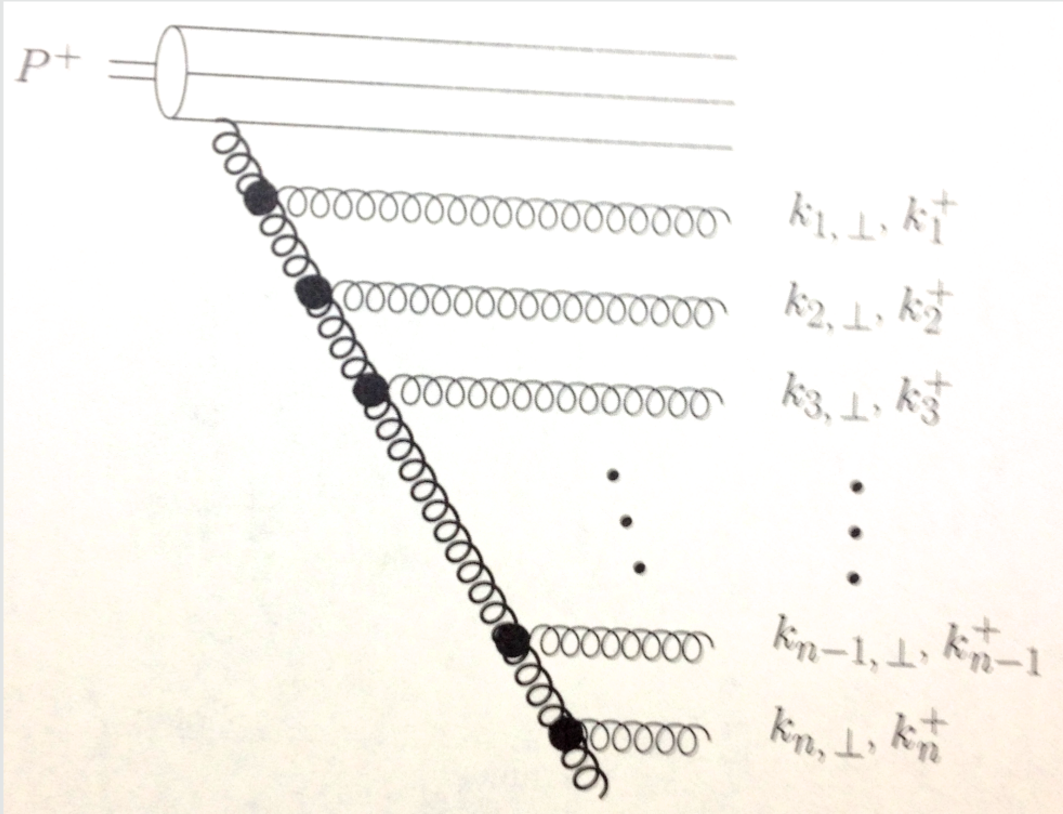
Radiated gluons act as sources for
the emissions of new gluons

DGLAP cascade



$$k_{\perp n}^2 \gg k_{\perp n-1}^2 \gg \cdots \gg k_{\perp 2}^2 \gg k_{\perp 1}^2$$

BFKL cascade



$$x_1^+ \gg x_2^+ \gg \cdots x_n^+$$

Growth of structure functions is tamed by non-linear contributions in evolution equations

[Gribov, Levin, Ryskin,83']

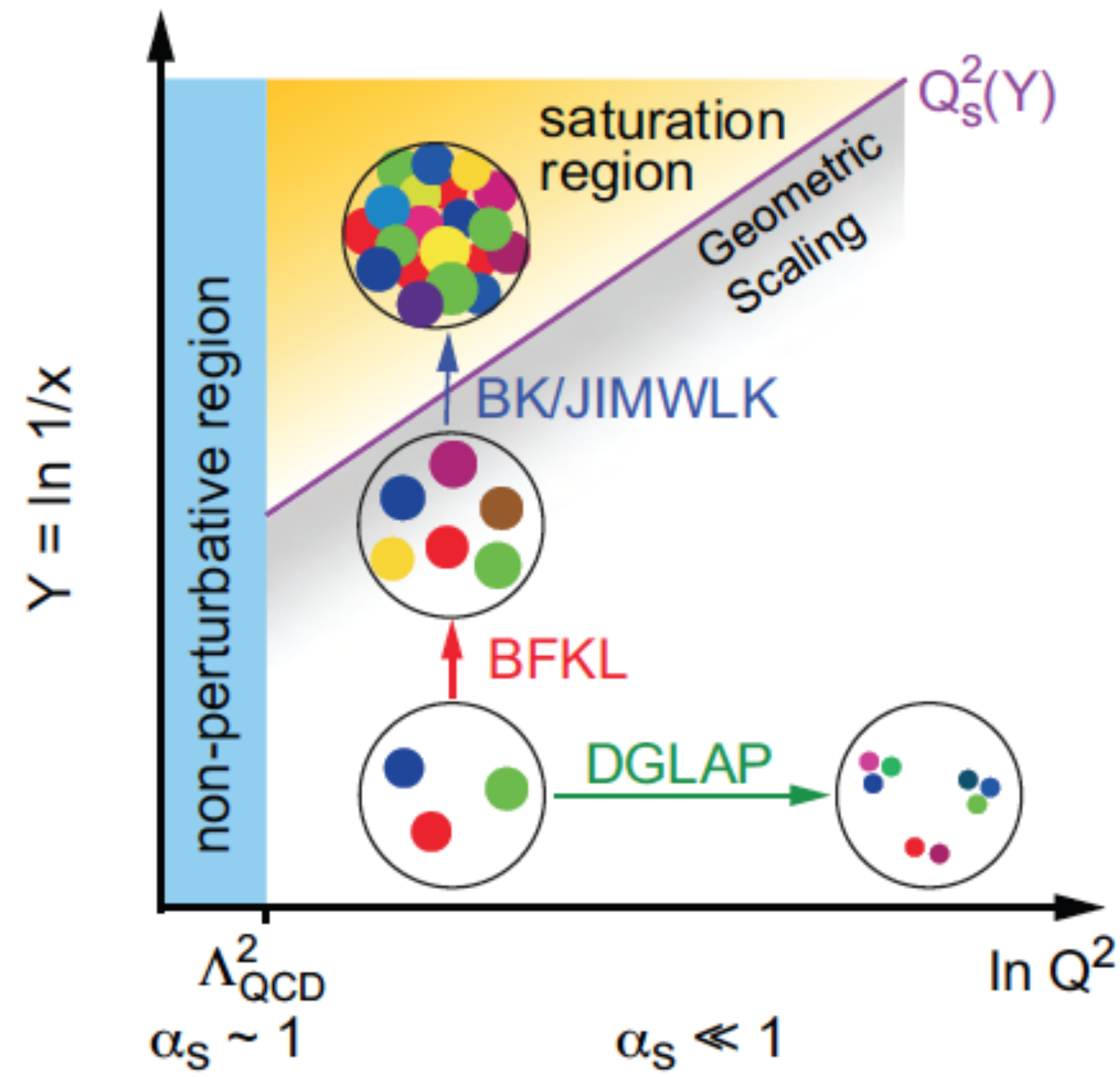
For instance,

$$\frac{\partial^2 \; xG(x,Q^2)}{\partial \ln(1/x)\,\partial \ln Q^2} = \bar{\alpha}_s \, xG(x,Q^2) - \bar{\alpha}_s^2 \, \frac{\pi^3}{2} \, \frac{[xG(x,Q^2)]^2}{R^2 Q^2}$$

[Gribov, Levin, Ryskin,83'- Mueller, Qiu, 86']

Emergence of a scale: the saturation momentum

$$Q_s^2 \sim \alpha_s \frac{xG(x,Q_s^2)}{\pi R^2} \qquad\qquad \frac{xG(x,Q_s^2)}{\pi R^2} \sim \frac{1}{\alpha_s}$$

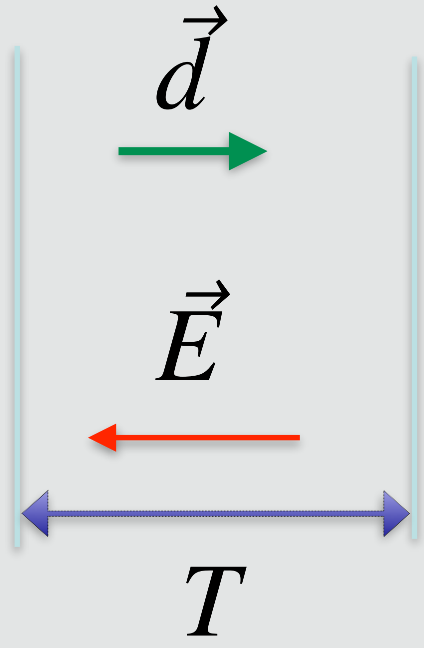

Y = ln 1/x
non-perturbative region
saturation region
$Q_s^2(Y)$
Geometric Scaling
BK/JIMWLK
BFKL
DGLAP
Λ^2_{QCD}
ln Q^2
$\alpha_s \sim 1$
$\alpha_s \ll 1$

Color dipoles
Wilson lines

Warm up exercise

Interaction of an electric dipole with a random field E

$$H = -\vec{d} \cdot \vec{E}$$



Evolution operator

$$U = \mathrm{e}^{-iHT} = \mathrm{e}^{idET}$$

Average over the random (Gaussian) distribution of E field

$$S = \mathrm{e}^{-\frac{1}{4}\frac{d^2}{r_s^2}} \qquad \frac{1}{r_s^2} = \langle E^2 T^2 \rangle$$

« Survival probability »

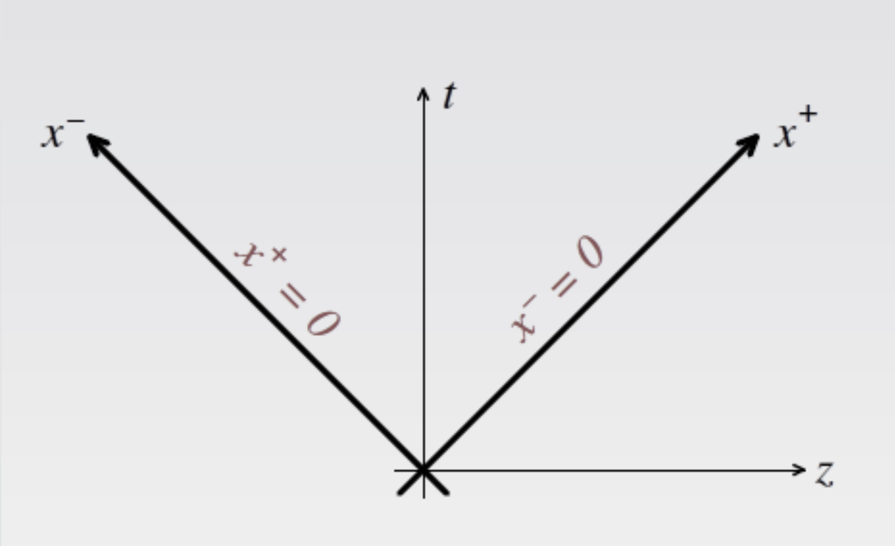
$$S^2 = \mathrm{e}^{-\frac{1}{2}\frac{d^2}{r_s^2}}$$

Eikonal propagation of the quark in the nucleus

$$U(\mathbf{x}_\perp) \equiv \mathcal{P} \exp\left[-ig \int_{-\infty}^{+\infty} dx^- A^+(x^-, \mathbf{x}_\perp)\right]$$

Dipole-nucleus scattering amplitude

$$S(\mathbf{b}, \mathbf{r}_\perp) = \frac{1}{N_c} \mathrm{Tr} \left\langle U(\mathbf{b} + \frac{\mathbf{r}_\perp}{2})\, U^\dagger(\mathbf{b} - \frac{\mathbf{r}_\perp}{2}) \right\rangle$$



$$x^\pm = \frac{x^0 \pm x^3}{\sqrt{2}}$$

Color dipole in eikonal approximation

S-matrix for a high energy quark moving in negative z-direction

$$U(\boldsymbol{x}_\perp) \equiv \mathcal{P} \exp\left[-ig \int_{-\infty}^{+\infty} dz^- A^+(z^-, \boldsymbol{x}_\perp)\right]$$

Dipole cross section

$$\sigma_{dip} = 2 \int d^2\boldsymbol{b}\,(1 - S(\boldsymbol{b}, \boldsymbol{r}_\perp))$$

$$S(\boldsymbol{b}, \boldsymbol{r}_\perp) = \frac{1}{N_c} \mathrm{Tr} \left\langle U(\boldsymbol{b} + \frac{\boldsymbol{r}_\perp}{2})\, U^\dagger(\boldsymbol{b} - \frac{\boldsymbol{r}_\perp}{2}) \right\rangle$$

For Gaussian fluctutations of the gauge field, one can calculate S

$$S(r_\perp) = {}^{-Q_s^2 r_\perp^2/4} \qquad Q_s^2 \propto \langle E(x_\perp)^2 \rangle \propto xG(x, 1/r_\perp^2)$$

Note « color transparency »: the scattering amplitude vanishes as $r_\perp \to 0$

« Black disk » limit when $r_\perp \gg 1/Q_s$

Dipole S-matrix in terms of Wilson lines (eikonal)

$$S(\mathbf{b},\mathbf{r}_\perp) = \frac{1}{N_c}\mathrm{Tr}\left\langle U(\mathbf{b}+\frac{\mathbf{r}_\perp}{2})\, U^\dagger(\mathbf{b}-\frac{\mathbf{r}_\perp}{2})\right\rangle$$

$$U(\mathbf{x}_\perp) \equiv \mathcal{P}\exp\left[-ig\int_{-\infty}^{+\infty} dx^- A^+(x^-,\mathbf{x}_\perp)\right]$$

Assume gaussian average

$$S(\mathbf{r}_\perp) = \exp\left\{-g^2 C_F \langle A^+(\mathbf{r}_\perp/2)A^+(-\mathbf{r}_\perp/2)\rangle\right\}$$

$$\langle A^+(\mathbf{r}_\perp/2)A^+(-\mathbf{r}_\perp/2)\rangle = \int \frac{d^2\mathbf{k}_\perp}{(2\pi)^2}\,\frac{1-\mathrm{e}^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp}}{\mathbf{k}_\perp^4}\mu_A^2(\mathbf{k})$$

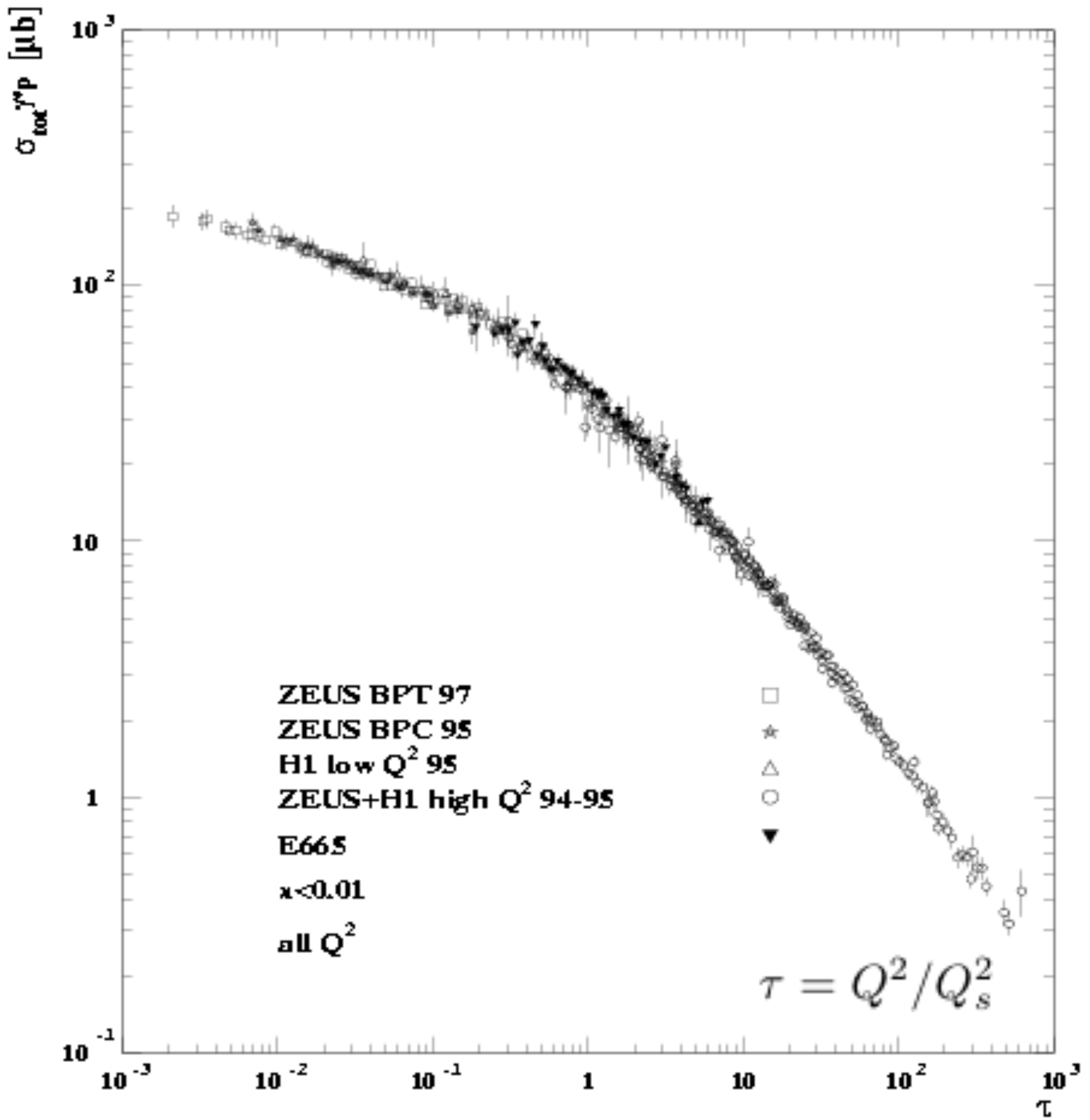
$$\langle A^+(\mathbf{r}_\perp/2)A^+(-\mathbf{r}_\perp/2)\rangle \approx \mu_A^2\frac{\mathbf{r}_\perp^2}{16\pi}\ln\frac{r_0^2}{r_\perp^2}$$

$$S(\mathbf{r}_\perp) = \exp\left\{-\frac{Q_s^2\mathbf{r}_\perp^2}{4}\right\} \qquad Q_s^2 = \alpha C_F \mu_A^2 \ln(r_0^2/r_\perp^2)$$

Geometrical scaling

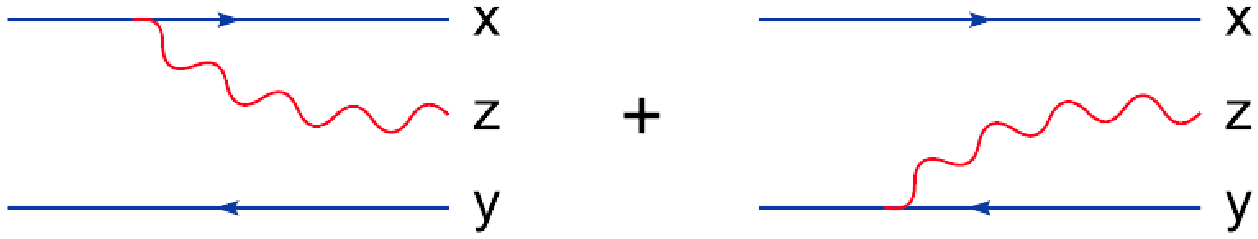
$$Q_s^2(x) \equiv Q_0^2 \left(\frac{x_0}{x}\right)^\lambda$$

$$\sigma(x, Q^2) = \sigma(Q^2/Q^2(x))$$



(Golec-Biernat,Kwiecinski,Stasto)

The evolution of the dipole amplitude



(B-JIMWLK)

$$\partial_Y \langle \mathrm{Tr}\left(U_x^\dagger U_y\right)\rangle_Y = -\frac{\alpha_s}{2\pi^2}\int d^2z\, \mathcal{K}_{xyz}\,\langle N_c \mathrm{Tr}\left(U_x^\dagger U_y\right) - \mathrm{Tr}\left(U_x^\dagger U_z\right)\mathrm{Tr}\left(U_z^\dagger U_y\right)\rangle$$

$$\mathcal{K}_{xyz} \equiv \frac{(x-y)^2}{(x-z)^2(y-z)^2}$$

(BK) $$\langle \mathrm{Tr}\left(U_x^\dagger U_z\right)\mathrm{Tr}\left(U_z^\dagger U_y\right)\rangle \approx \langle \mathrm{Tr}\left(U_x^\dagger U_z\right)\rangle\langle \mathrm{Tr}\left(U_z^\dagger U_y\right)\rangle$$

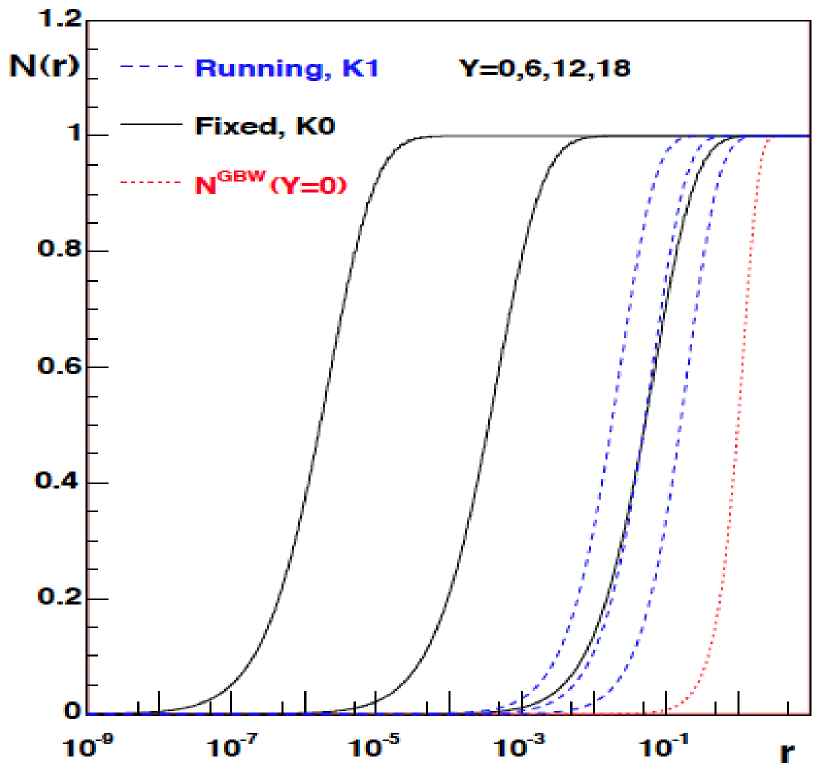
$$S = 1 - N$$

$$\partial_Y N_{xy} = -\frac{\alpha_s N_c}{\pi}\int \frac{d^2z}{2\pi}\mathcal{K}_{xyz}\,\left(N_{xz} + N_{zy} - N_{xy} - N_{xy}N_{zy}\right)$$

The 'saturation front' and its universal behavior

Travelling wave solutions [Meunier, Peschanski, 05]

$$N(r, Y) = N(r - r_s(Y))$$



'Geometrical scaling' naturally emerges

Analogy with reaction difffusion processes

$$\partial_Y T(\rho, Y) = \partial_\rho^2 T(\rho, Y) + T(\rho, Y) - T^2(\rho, Y)$$

Classical fields

CGC

Averaging over color field of the nucleus

$$\frac{1}{N_c}\mathrm{Tr}\left\langle U(\mathbf{b}+\frac{\mathbf{r}_\perp}{2})\, U^\dagger(\mathbf{b}-\frac{\mathbf{r}_\perp}{2})\right\rangle_Y$$

During interaction process, the field A of the target is frozen (separation of scales - adiabatic approximation)

$$\langle \cdots \rangle_Y = \int \mathcal{D}A \; |\Phi_Y[A]|^2 \; \langle A| \cdots |A\rangle$$

Fields are created by (frozen)sources. Fields are obtained from Yang-Mills equations

$$[D_\mu, F^{\mu\nu}] = J^\nu \qquad J^\mu(x^-, x_\perp) = \delta^{\mu +}\rho(x^-, x_\perp)$$

Emphasis is put on small x part of the wave function (strong sources)

More conventional notation (fields -> color charges)

$$|\Phi_Y[A]|^2 \leftrightarrow W_Y[\rho]$$

Evolution equations (JIMWLK, BK) may be viewed as non linear equations for $W_Y[\rho]$

Emphasis is put on color charge distributions and their correlations

MV model

$$\langle \rho^a(x^-, x_\perp)\rho^b(y^-, y_\perp)\rangle = \delta_{ab}\delta(x^- - y^-)\delta^{(2)}(x_\perp - y_\perp)\mu^2(x^-)$$

$$f_A(\boldsymbol{k}_\perp) = \frac{1}{\alpha N_c}\int d^2\boldsymbol{r} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_\perp} \frac{1 - e^{-Q_s^2 \boldsymbol{r}_\perp^2/4}}{\pi \boldsymbol{r}_\perp^2}$$

$$Q_s^2 = \frac{4\pi^2\alpha}{N_c^2 - 1} n(b) x G(x, Q_s^2)$$

So, what is the color glass condensate?

- Evolution equation
- Attempt to calculate ‘wave functions’ from first principles

Wilson line operators and the U–representation

Calculate change in expectation values, $\partial_\tau \langle \mathrm{tr}(U_{\boldsymbol{x}}^\dagger U_{\boldsymbol{y}})\rangle_\tau$, rather than the expectation values themselves. Focus on the evolution of the dipole

$$\partial_\tau \langle \mathrm{tr}(U_{\boldsymbol{x}}^\dagger U_{\boldsymbol{y}})\rangle_\tau =$$

$$-\frac{\alpha_s}{2\pi^2}\int d^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\left\langle N_c \mathrm{tr}(U_{\boldsymbol{x}}^\dagger U_{\boldsymbol{y}}) - \mathrm{tr}(U_{\boldsymbol{x}}^\dagger U_{\boldsymbol{z}})\mathrm{tr}(U_{\boldsymbol{z}}^\dagger U_{\boldsymbol{y}})\right\rangle_\tau$$

$$\langle U_{\boldsymbol{x}_1}^{(\dagger)} \ldots U_{\boldsymbol{x}_n}^{(\dagger)}\rangle_\tau = \int [\mathrm{d}\mu(U)]\; U_{\boldsymbol{x}_1}^{(\dagger)} \ldots U_{\boldsymbol{x}_n}^{(\dagger)}\; Z_\tau[U]$$

$$\boxed{\partial_\tau Z_\tau[U] \;=\; \frac{1}{2}\,\nabla_{\boldsymbol{x}}^a\, \chi_{\boldsymbol{x}\boldsymbol{y}}^{ab}[U]\,\nabla_{\boldsymbol{y}}^b\, Z_\tau[U]}$$

$$\chi_{\boldsymbol{x}\boldsymbol{y}}^{ab}[U] \equiv \frac{1}{\pi}\int \frac{d^2\boldsymbol{z}}{(2\pi)^2}\,\mathcal{K}_{\boldsymbol{x}\boldsymbol{y}\boldsymbol{z}}\left[(1-U_{\boldsymbol{x}}^\dagger U_{\boldsymbol{z}})(1-U_{\boldsymbol{z}}^\dagger U_{\boldsymbol{y}})\right]^{ab}$$

$$\mathcal{K}_{\boldsymbol{x}\boldsymbol{y}\boldsymbol{z}} \equiv \frac{(\boldsymbol{x}-\boldsymbol{z})\cdot(\boldsymbol{y}-\boldsymbol{z})}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}$$

The Colour Glass Condensate

A model for the hadron wavefunction $\longrightarrow$ explicit calculation of expectation values. Soft gluons (xP^+) treated as classical fields. Fast partons ($k^+ \gg xP^+$) create a source $\rho_a(x^-, \boldsymbol{x})$ for the soft fields.

$\rho_a(x^-, \boldsymbol{x})$ **is a random variable**

$$(D_\nu F^{\nu\mu})_a(x) \; = \; \delta^{\mu +}\rho_a(x^-, \boldsymbol{x})$$

Renormalization group procedure $\rho \longrightarrow \rho + \delta\rho$

$$\langle \delta\rho \rangle \sim \sigma \qquad\qquad \langle \delta\rho\delta\rho \rangle \sim \chi \qquad \sigma = \frac{1}{2}\frac{\delta\chi}{\delta\rho}$$

$$\langle \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}}) \rangle_\tau = \int [\mathrm{d}\rho] \, W_\tau[\rho] \, \langle \mathrm{tr}(U^\dagger_{\boldsymbol{x}}[\rho] U_{\boldsymbol{y}})[\rho] \rangle$$

$$\boxed{\partial_\tau W_\tau[\rho] \; = \; \frac{1}{2}\frac{\delta}{\delta\rho^a_\tau(\boldsymbol{x})}\chi^{ab}_{\boldsymbol{xy}}[\rho]\frac{\delta}{\delta\rho^b_\tau(\boldsymbol{y})} W_\tau[\rho]}$$

Empirical evidences

Empirical evidences

- Geometrical scaling (DIS, photon-A,pp, etc…)
- Multiplicity in HI collisions, energy dependence, centrality dependence
- Limiting fragmentation
- Long range rapidity correlations ('ridge', in AA, in pp)
- Forward rapidity phenomena (disappearance of Cronin peak, disappearance of dijet correlations)

For a recent review, see Albacete, Marquet in arXiv:1401.4866

Phenomenology based on a few ingredients

NB. i) Phenomenology is blind to many details of the theory.
ii) Many features hold only in asymptotic regimes

Saturation momentum

$$Q_s^2 = Q_0^2 \left(\frac{x}{x_0}\right)^\lambda \qquad Q_0^2(b) = Q_0^2(0) T_A(\boldsymbol{b}) \qquad T_A(\boldsymbol{b}) = \int dz \rho(\boldsymbol{b}, z)$$

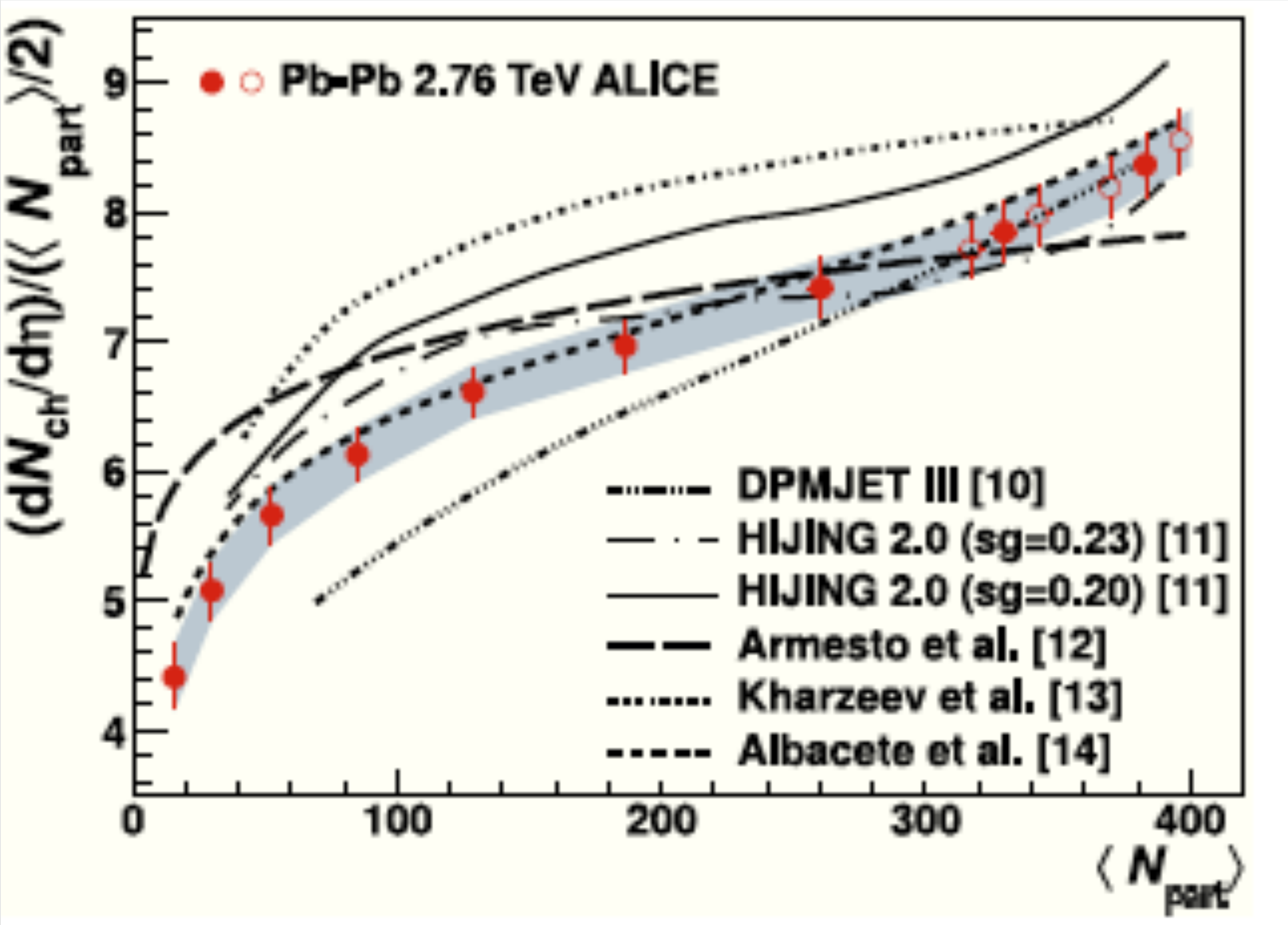
Running coupling

$$\alpha_s = \alpha_s(Q_s)$$

kT factorization

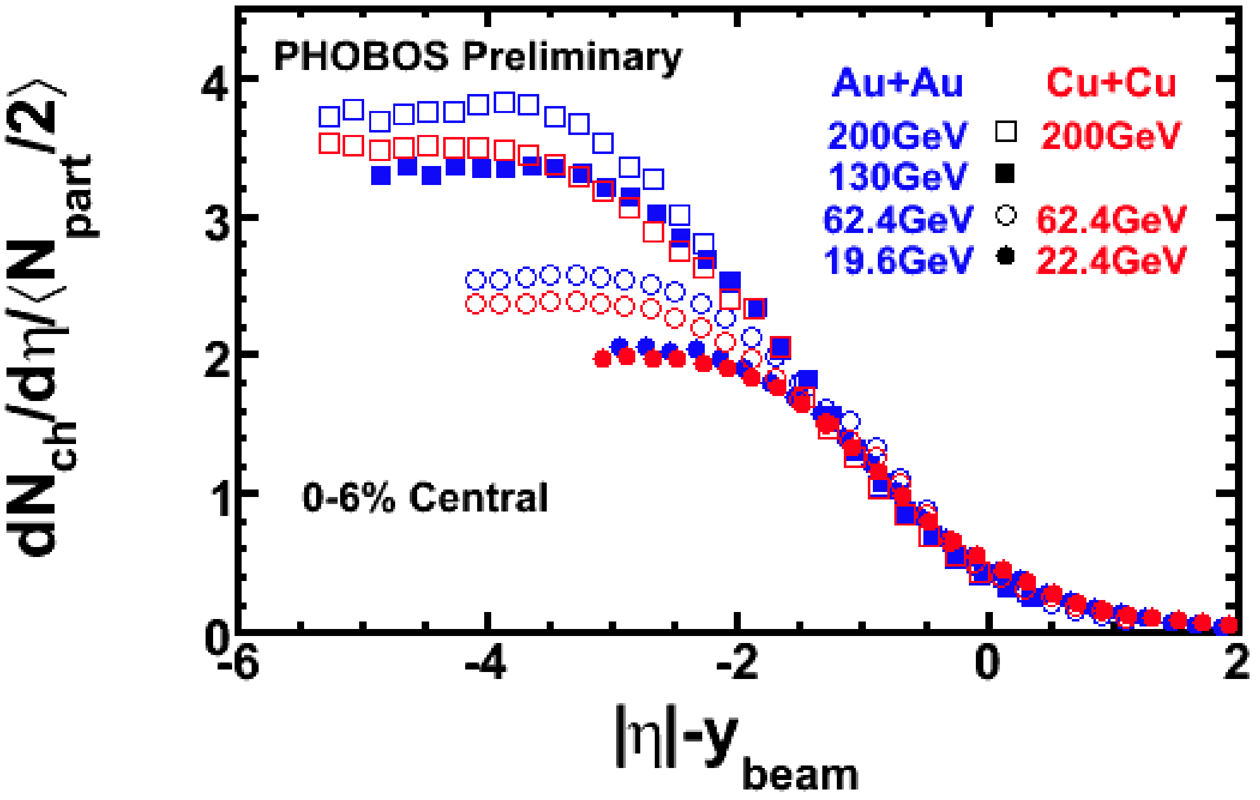
$$\frac{dN}{dy d^2\boldsymbol{p}_\perp d^2\boldsymbol{b}} = \frac{1}{2\pi^4 C_F} \frac{\alpha_s}{\boldsymbol{p}_\perp^2} \int \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} \phi_A(x_1, k_\perp) \phi_B(x_2, |\boldsymbol{p}_\perp - \boldsymbol{k}_\perp|)$$

Evolution equation (BK or improved versions, rcBK, etc)



Pb-Pb 2.76 TeV ALICE
DPMJET III [10]
HIJING 2.0 (sg=0.23) [11]
HIJING 2.0 (sg=0.20) [11]
Armesto et al. [12]
Kharzeev et al. [13]
Albacete et al. [14]
(dNch/dη)/(⟨Npart⟩/2)
⟨Npart⟩
0
100
200
300
400
4
5
6
7
8
9

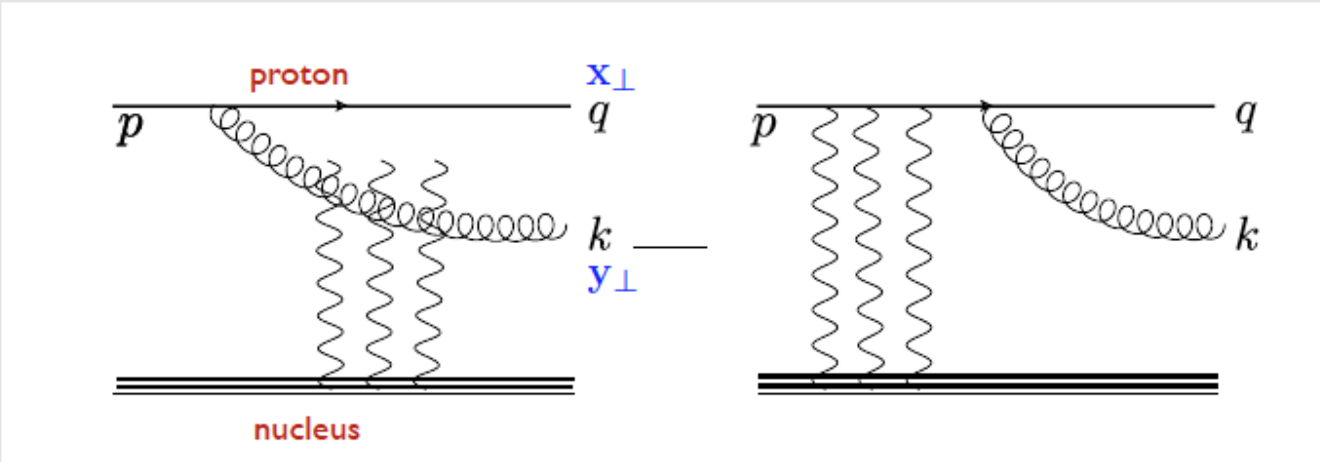
Limiting fragmentation



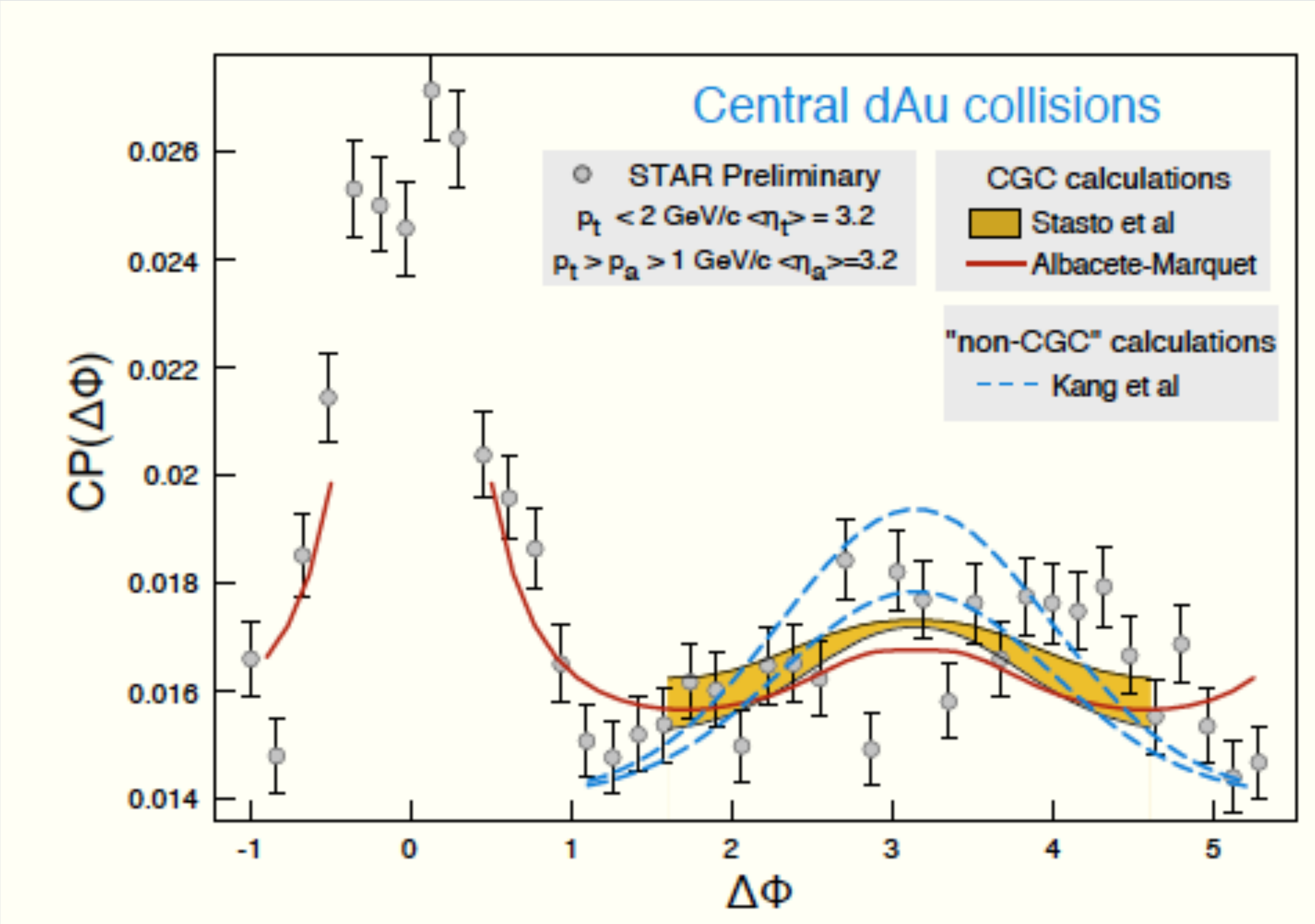
Forward correlation in di-hadron production

$$p + A \to h_1 h_2 X$$

$$x_i = \frac{|p_{i\perp}|}{\sqrt{s_{NN}}} \qquad x_A = x_1 \mathrm{e}^{-2y_1} + x_2 \mathrm{e}^{-2y_2}$$



$$qA \to qgX$$


Central dAu collisions
STAR Preliminary
p_t < 2 GeV/c <η_t> = 3.2
p_t > p_a > 1 GeV/c <η_a>=3.2
CGC calculations
Stasto et al
Albacete-Marquet
"non-CGC" calculations
Kang et al
CP(ΔΦ)
ΔΦ
0.026
0.024
0.022
0.02
0.018
0.016
0.014
-1
0
1
2
3
4
5

The calculation involves complicated correlators of Wilson lines

$$S^{(6)}(x_\perp, x'_\perp, y_\perp, y'_\perp) = \Big\langle -\frac{1}{N_c(N_c^2-1)} \,\mathrm{tr}\left\{U(x_\perp)U^\dagger(x'_\perp)\right\}$$
$$+\frac{1}{N_c^2-1}\,\mathrm{tr}\left\{U(y_\perp)U^\dagger(y'_\perp)\right\}\,\mathrm{tr}\left\{U(x_\perp)U^\dagger(x'_\perp)U(y_\perp)U^\dagger(y'_\perp)\right\}\Big\rangle_Y$$

- intrinsically difficult (can be simplified a bit in large Nc)
- initial conditions ?