Matrix combination of BFKL and DGLAP

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Work with M. Ciafaloni, D. Colferai and A. Stasto [arXiv:0707.1453]

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Introduction

This talk is a progress report on a long-term project to put together *DGLAP* and the linear regime of *BFKL* evolution, including higher order and running-coupling corrections.

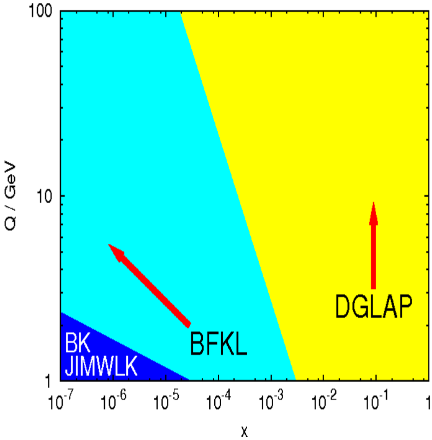
Main groups active:

- Altarelli, Ball, Forte (+ Falgari, Marzano) aka ABF
- Ciafaloni, Colferai, GPS, Staśto aka CCSS

+ Thorne & White

Motivation

When looking at proton structure we can establish different evolution regimes:



But:

- Regions of validity not clearly delimited
- Higher orders of DGLAP contaminated by leading BFKL:

$$P_{gg}(x) \simeq \frac{\bar{\alpha}_s}{x} + \bar{\alpha}_s^4 \frac{\zeta(3)}{3} \frac{\ln^3 x}{x} + \dots$$

- Higher orders of BFKL contaminated by leading DGLAP:

$$K(k, k') \simeq \bar{\alpha}_s - \bar{\alpha}_s^2 \frac{11}{12} \ln \frac{k^2}{k'^2} + \dots$$

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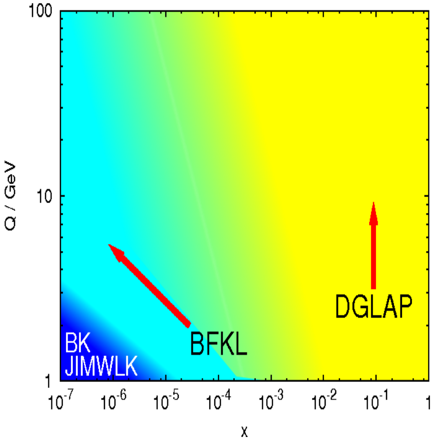

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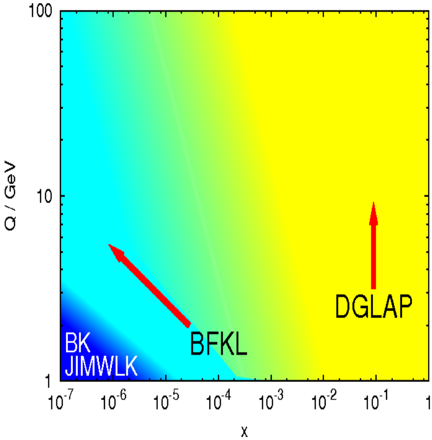

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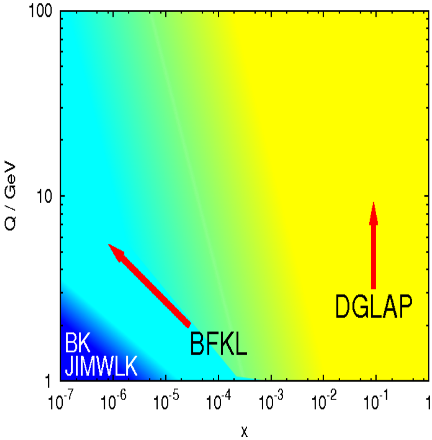

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DGLAP, BFKL (fixed coupling)

DGLAP

Integro(x)–differential(Q^2) eq^n for *integrated* gluon dist., g:

$$\frac{dg(x,Q^2)}{d\ln Q^2} = \int \frac{dz}{z} P_{gg}(z) g(\frac{x}{z}, Q^2)$$

BFKL

Integro(k)–differential(x) eq^n for *unintegrated* gluon dist., G:

$$\frac{dG(x,k^2)}{d\ln 1/x} = \int \frac{dk'^2}{k'^2} K(k/k') G(x, k'^2)$$

k, Q are transverse scales; x is longitudinal mom. fraction

$xg(x,Q^2) = \int^Q d^2k G(x,k^2)$

Both DGLAP and BFKL relate $\perp$ structure to long. structure:

- given long. struct. DGLAP gives you $\perp$ struct. evolution
- given $\perp$ struct. BFKL gives you long. struct. evolution

When calculated at all orders they must encode the same physics.

Inevitable that one contaminated by other at fixed order

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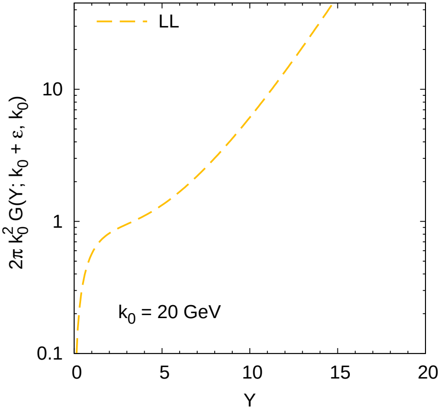
NLL Green function solution

If DGLAP contaminates BFKL does it matter? Can we not just take the perturbative expansion? Try LL, then NLL BFKL.



Choices that formally only affect NNLLx:

- scale of α_s
- 'energy-scale' s_0 $(Y = \ln s/s_0)$.

lead to completely different answers

Source of instability is presence in NLL BFKL of a truncated subset of DGLAP. Only way to get stability is to include full DGLAP.

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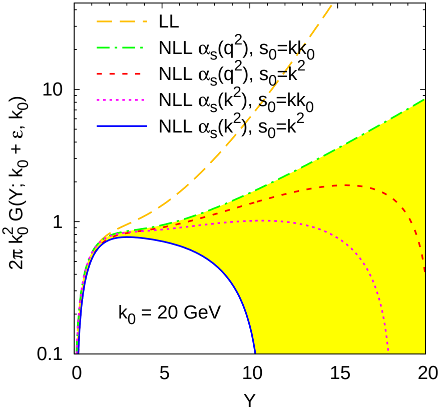

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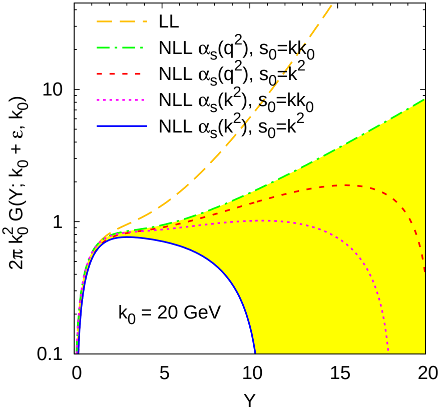

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Merging BFKL & DGLAP

Long history of work on merging leading BFKL and DGLAP.

CCFM '88; Lund group ∼ '95; Durham-Cracow group ∼'95;

Two approaches have been used in order to combine BFKL and DGLAP *including higher orders:*

- Establish all-order relation (*duality relation*) between splitting functions (DGLAP) and evolution kernel (BFKL). Use that to simultaneously construct splitting functions consistent with BFKL kernel and vice-versa.

  Altarelli, Ball & Forte '99–

- Establish a more general equation that embodies both BFKL and DGLAP (*double-integral equation*):

$$G(x,k^2) = G_0(x,k^2) + \int dz \int dk'^2 \frac{dk'^2}{k'^2} K(z,k,k') G(x/z,k'^2)$$

  From that, deduce *effective* splitting function and BFKL kernel.

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Pure glue case, LLx+LO

Write Kernel as power series in α_s: $K = \sum_{n=0} \hat{\alpha}^n K_n$ $\hat{\alpha} = \alpha_s/2\pi$

First order (*LLx-LO*) has two parts:

$$K_0(\gamma, \omega) = \underbrace{\frac{2C_A}{\omega}\chi_0^\omega(\gamma)}_{\text{BFKL (LL}x\text{)}} + \underbrace{\left[\Gamma_{gg,0}(\omega) - \frac{2C_A}{\omega}\right]\chi_c^\omega(\gamma)}_{\text{finite-}x\text{ DGLAP (LO)}}$$

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BFKL piece has usual transverse structure with *kinematic constraint*

$$\chi_0^\omega(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1+\omega-\gamma)$$

Note symmetry $\gamma \leftrightarrow 1-\gamma+\omega$

Multiplied by $\alpha_s(q^2)$, $\vec{q} = \vec{k} - \vec{k}'$

DGLAP remainder piece has a *collinear kernel*:

$$\chi_c^\omega(\gamma) = \frac{1}{\gamma} + \frac{1}{1+\omega-\gamma}$$

Multiplied by $\alpha_s(k_>^2)$

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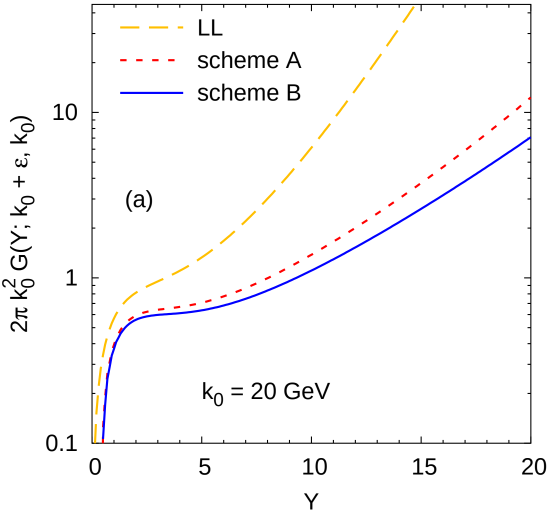
Next order (*NLx-NLO*) also has two parts:

$$K_1(\gamma,\omega) = \frac{(2C_A)^2}{\omega}\,\tilde{\chi}_1^\omega(\gamma) + \tilde{\Gamma}_{gg,1}(\omega)\,\chi_c^\omega(\gamma)$$

with $\tilde{\chi}_1$ and $\tilde{\Gamma}_{gg,1}(\omega)$ adjusted so as to reproduce NLx BFKL and NLO DGLAP.

Green fn. from improved kernel



First tried in '03, without NLO DGLAP piece.

NLx-LO

Two schemes, to estimate degree of stability

- scheme A violates mom. sum-rule at $\mathcal{O}\left(\alpha_s^2\right)$
- scheme B satisfies it at all orders

Solve double-integral eq^n with each.

Different schemes → similar results

Green function ⇒ effective P_{gg}

Construct a gluon density from Green function (take $k \gg k_0$):

$$xg(x, Q^2) \equiv \int^Q d^2k \; G^{(\nu_0 = k^2)}(\ln 1/x, k, k_0)$$

Numerically solve equation for effective splitting function, $P_{gg,\mathrm{eff}}(z, Q^2)$:

$$\frac{dg(x, Q^2)}{d \ln Q^2} = \int \frac{dz}{z} P_{gg,\mathrm{eff}}(z, Q^2) \, g\left(\frac{x}{z}, Q^2\right)$$

Factorisation

- Splitting function:

  red paths

- Green function:

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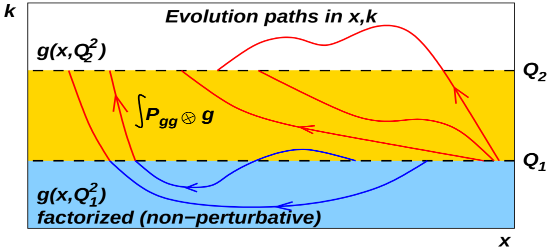
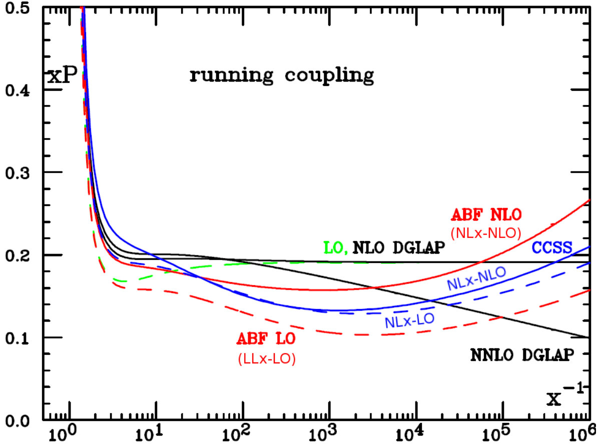


One channel P_{gg}: ABF v. CCSS



Altarelli, Ball & Forte have also calculated effective P_{gg}:

- similar physical ingredients
- completely different 'implementation'

Main features similar between CCSS & ABF.

In particular splitting-fn has **dip** at $x \sim 10^{-3}$.

Two channels

BFKL is naturally single-channel — Only gluon production has $1/x$ divergence

DGLAP is multi-channel — Quarks and gluons both have collinear divergences

So far we had *ignored the multi-channel aspect,* for simplicity. But:

- If we are to use small-x resummed splitting functions, we need the whole singlet matrix
- Including quarks in evolution may provide a convenient way of resumming collinear logs in impact factors

Generalise double-integral eq^n to two channels

Add flavour indices to Green function and kernel

$$G_{ab}(x,k^2,k_0^2) = \delta^2(k-k_0)\delta_{ab} + \int dz \int dk'^2 \frac{dk'^2}{k'^2} K_{ac}(z,k,k') G_{cb}(x/z, k'^2, k_0^2)$$

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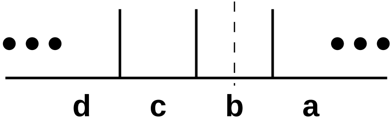
Symmetry and subtleties



Want to encode two strongly ordered collinear limits

DGLAP limit $\ldots\Gamma_{dc}\,\Gamma_{cb}\,\Gamma_{ba}\,\ldots$ $\begin{cases} x_d < x_c < x_b < x_a \\ k_{t\,d} \gg k_{t\,c} \gg k_{t\,b} \gg k_{t\,a} \end{cases}$

anti-DGLAP limit $\ldots\Gamma_{cd}\,\Gamma_{bc}\,\Gamma_{ab}\,\ldots$ $\begin{cases} \bar{x}_d > \bar{x}_c > \bar{x}_b > \bar{x}_a \\ k_{t\,d} \ll k_{t\,c} \ll k_{t\,b} \ll k_{t\,a} \end{cases}$

$$= \ldots(\Gamma^T)_{dc}\,(\Gamma^T)_{cb}\,(\Gamma^T)_{ba}\,\ldots\,.$$

Suggests sym. $K(\gamma,\omega) = K^T(1+\omega-\gamma,\omega)$. But this $\to$ spurious colour & $1/\omega$ structures, e.g. $\alpha_s^2 C_F^2/\omega^2$ for $g \to q \to g$, in non-ordered limits.

DGLAP attaches $1/\omega$ and colour sum to leg with higher p_t
BFKL attaches them to left-hand leg — **inconsistent**

Matrix requirements

Sensibleness requirement on matrix formulation.

Use similarity transform S to reattach colour and $1/\omega$ factors in anticollinear limit, so as to restore compatibility between DGLAP and BFKL. Resulting symmetry is

$$\mathcal{K}(1+\omega-\gamma,\omega) = S(\omega)\mathcal{K}^T(\gamma,\omega)S^{-1}(\omega)\,.$$

Choose S, for convenience, such that

$$\mathcal{K}^T(\gamma,\omega) = S(\omega)\mathcal{K}^T(\gamma,\omega)S^{-1}(\omega) \implies \mathcal{K}(1+\omega-\gamma,\omega) = \mathcal{K}(\gamma,\omega)$$

Other requirements

- K_{qq}, K_{qg} should be free of $1/\omega$ divergences at all orders
- K_{gq}, K_{gg} may at most have $1/\omega$ divergences
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Matrix kernel

Structure quite similar to single-channel; LLx-LO is:

$$\mathcal{K}_0(\gamma,\omega) = \begin{pmatrix} \Gamma_{qq,0}(\omega)\chi_c^\omega(\gamma) & \Gamma_{qg,0}(\omega)\chi_c^\omega(\gamma) + \Delta_{qg}(\omega)\chi_{\mathrm{ht}}^\omega(\gamma) \\ \Gamma_{gq,0}(\omega)\chi_c^\omega(\gamma) & \Gamma_{gg,0}(\omega)\chi_c^\omega(\gamma) + \frac{2C_A}{\omega}\left[\chi_0^\omega(\gamma) - \chi_c^\omega(\gamma)\right] \end{pmatrix}$$

Note $\Delta_{qg}(\omega)$ term: allows one to set *factorisation scheme* at NLO, by modifying the *higher-twist* part of the $\mathcal{K}_{qg}$ kernel.

Without having to add α_s^2/ω term to $\mathcal{K}_{1,qg}$

NB: We choose $\overline{\mathrm{MS}}$

Higher orders:

- Add on $\mathcal{K}_1(\gamma,\omega)$ to get NLx-NLO.
- put in extra higher-twist piece in $\mathcal{K}_0(\gamma,\omega)$ to get α_s^3/ω^2 scheme-dependent terms (NLx-NLO$^+$).

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  Based on study of P_{gg}, CCSS '06

- Formalism 'predicts' that at NLx accuracy, at NNLO

$$\Gamma_{gq,2}^{\mathrm{NL}x} = \frac{C_F}{C_A}\Gamma_{gg,2}^{\mathrm{NL}x}$$

But true $\overline{\text{MS}}$ [MVV '04] result differs by an N_c-suppressed term

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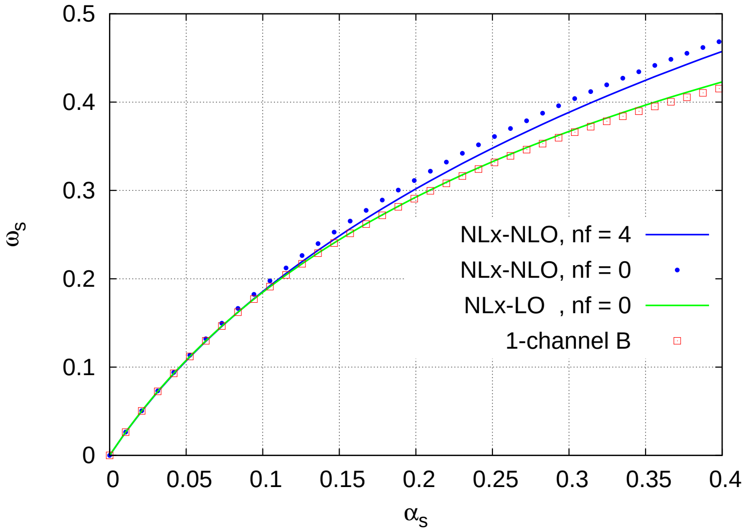
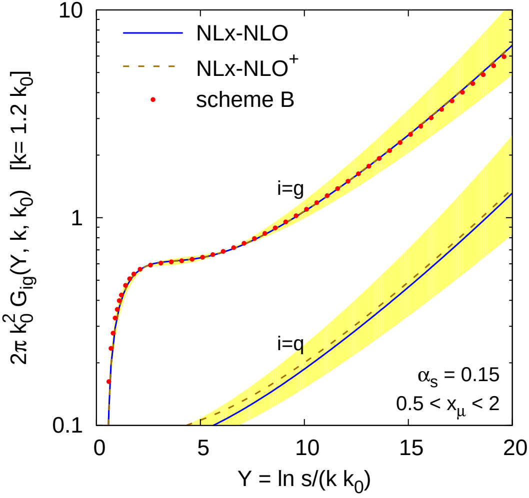
Intercept at fixed coupling

Power of growth of cross-sections and splitting functions at fixed coupling. Rather similar to 2003 results:



Green function



Green function for gluon is very similar to 2003 results.

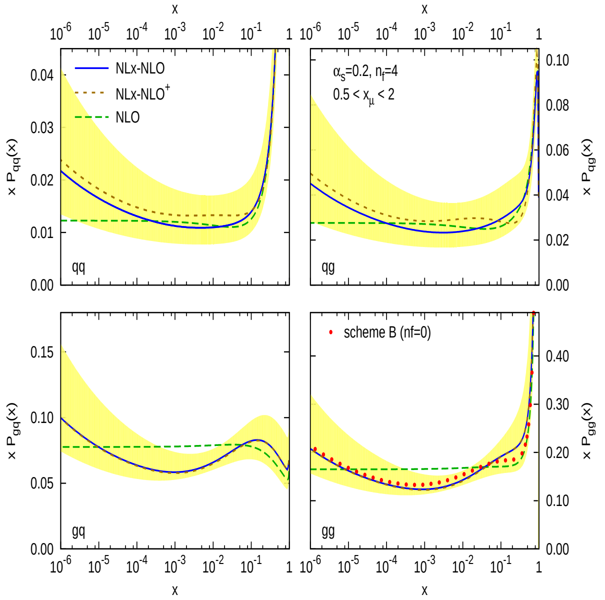
Scale uncertainties (band) under control

Additionally *generate quark component*, with same power-growth, but suppressed by $\sim \alpha_s$.

Scale uncertainties larger — radiative generation

NNLO part of NLx scheme terms (NLO$^+$) have little impact.

Splitting functions



In *gg* channel results again similar to those from 2003

gq channel rather similar to *gg*

Both have **dip** at $x \sim 10^{-3}$

qq and *qg* channels have barely any dip, and large scale uncertainties — NLx is first order of generation of small-x quarks.

Conclusions, outlook

- Have matrix double integral equation that contains both NLx BFKL and NLO DGLAP in $\overline{\text{MS}}$ scheme.
- From it one can deduce Green functions and matrix of effective small-x resummed splitting functions.
- Gluon-channel results agree with earlier resummations, now also get full singlet matrix.

Many options open for future

- providing splitting functions in convenient form for general use
- understanding what happens at NNLO
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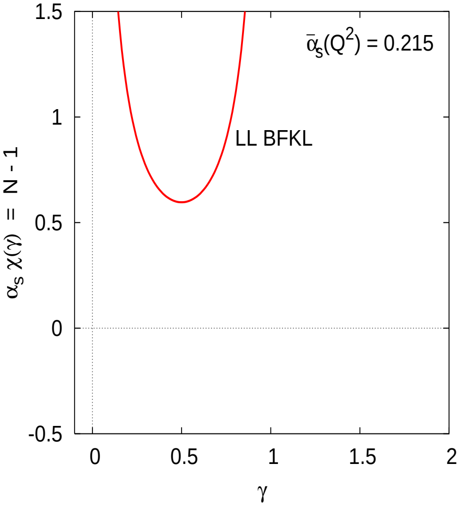
EXTRAS

Building up the kernel...



Examine $\bar{\alpha}_s \chi(\gamma)$

minimum = BFKL power

$$\chi(\gamma) = \underbrace{\chi_0(\gamma)}_{LL} + \underbrace{\bar{\alpha}_s \chi_1(\gamma)}_{NLL} + \dots$$

- NLL terms *pathologically large.*
  minimum → max. (unstable)
  oscillating X-sctns, ...
- Culprit: ⊥ DGLAP logs
  $$\frac{\bar{\alpha}_s}{\gamma} - \frac{11}{12}\frac{\bar{\alpha}_s^2}{\gamma^2} + \dots$$
  $[\gamma^{-1} \Leftrightarrow \ln Q^2]$
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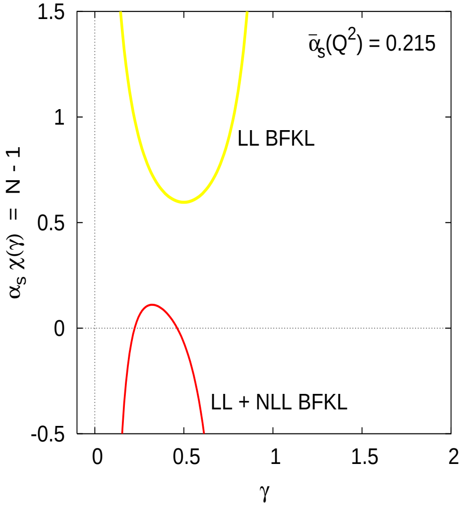

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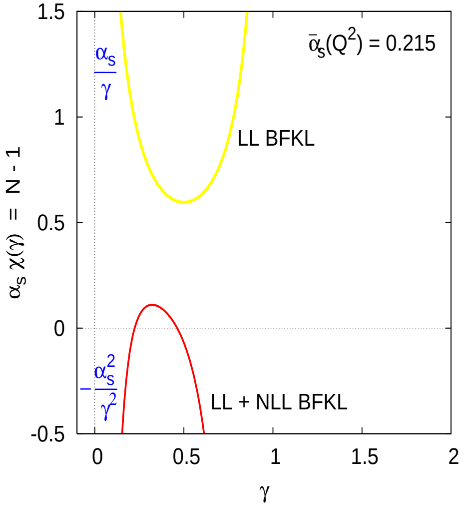

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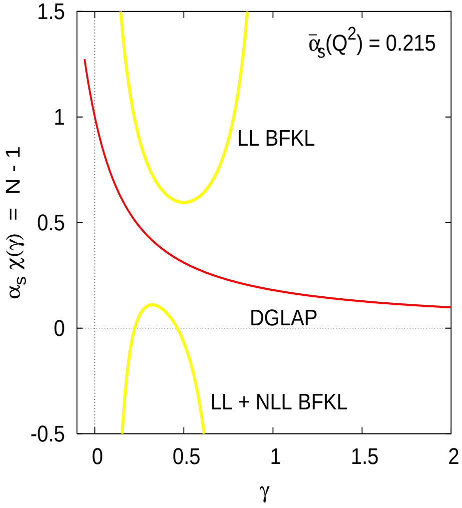

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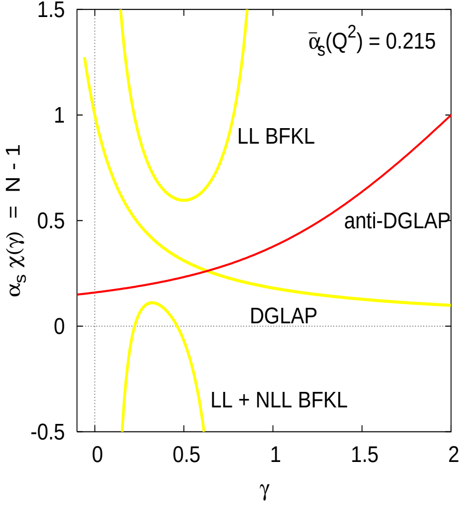

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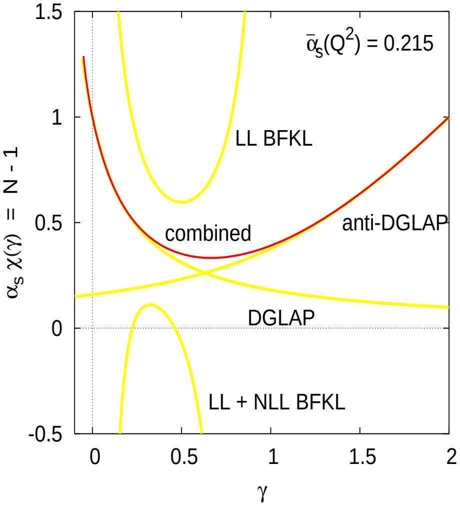

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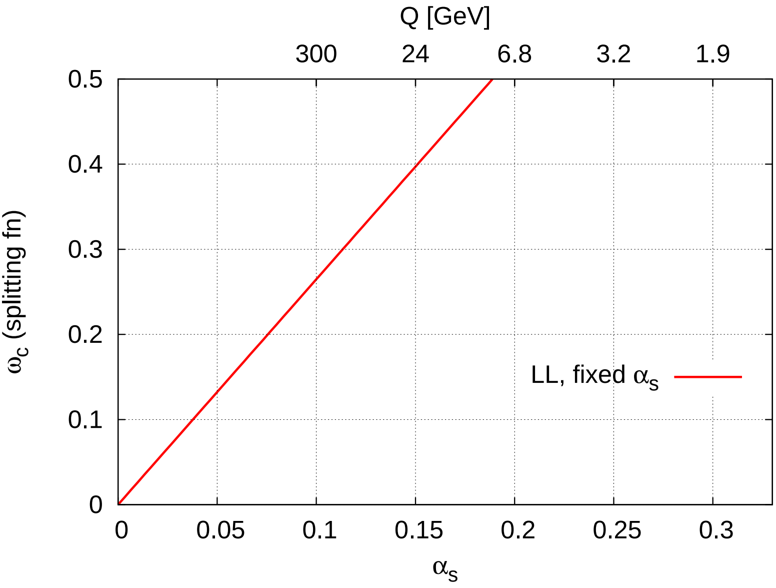
Assemble all constraints:

stable, sensible kernel

Ciafaloni, Colferai, GPS & Staśto;
Altarelli, Ball & Forte; '99–'05

ω_c vs. α_s



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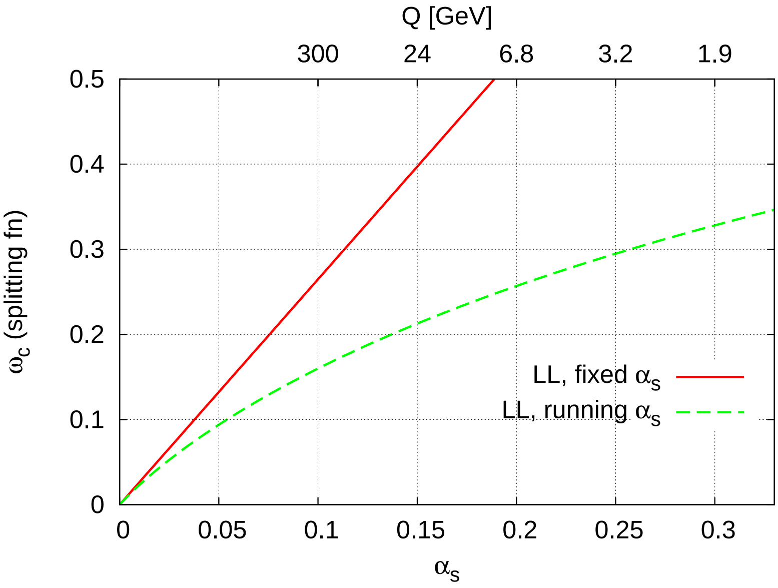

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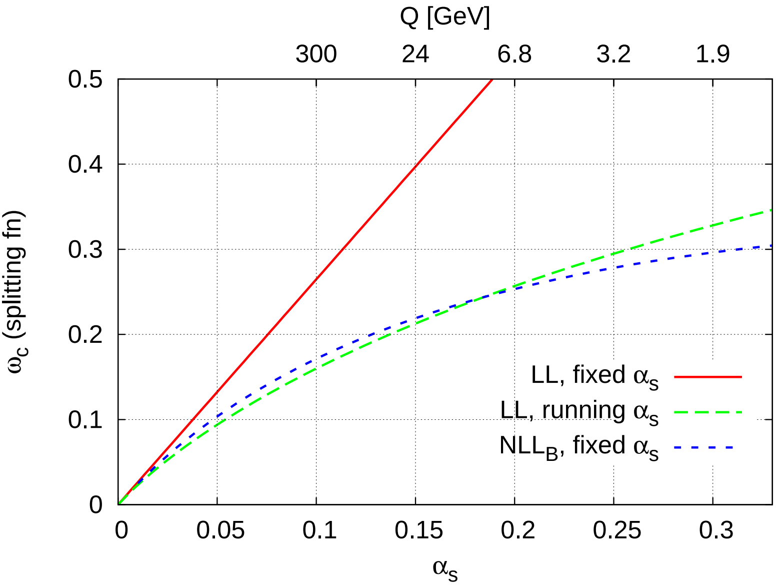

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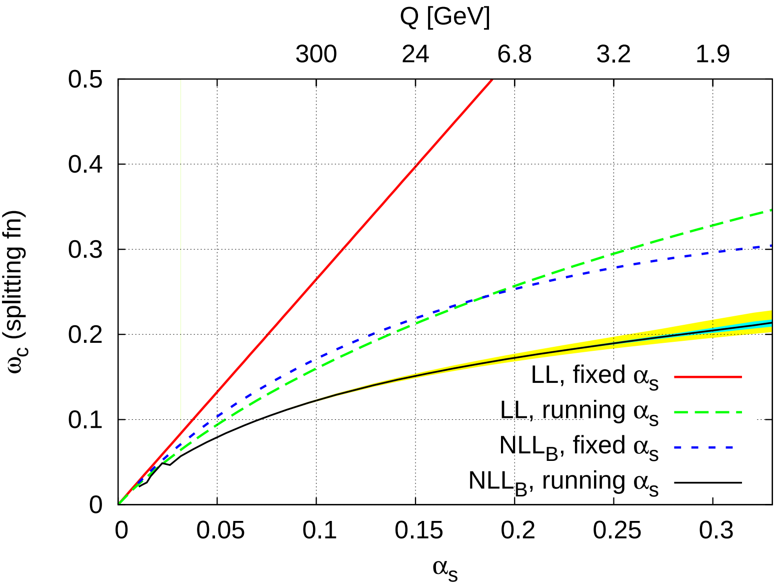

Similarity transforms

$$S = \begin{pmatrix} 2n_f N_c f_q(\omega) & 0 \\ 0 & (N_c^2 - 1) f_g(\omega) \end{pmatrix},$$

$$\overline{\Gamma} = S \Gamma^T S^{-1} = \begin{pmatrix} \Gamma_{qq} & \frac{n_f}{C_F} \frac{f_q(\omega)}{f_g(\omega)} \Gamma_{gq} \\ \frac{C_F}{n_f} \frac{f_g(\omega)}{f_q(\omega)} \Gamma_{qg} & \Gamma_{gg} \end{pmatrix}.$$

$$\mathcal{K} \simeq \frac{\Gamma}{\gamma} + \frac{\overline{\Gamma}}{1 + \omega - \gamma},$$

$$f_q(\omega) = \frac{2 T_R}{\omega + 3} \quad \Longrightarrow \quad \overline{\Gamma} = \Gamma,$$

Higher-order kernel

$$\mathcal{K}(\alpha_s, \gamma, \omega) \equiv \sum_{n,m,p=0}^{\infty} {}_p\mathcal{K}_n^{(m)} \, \hat{\alpha}^{n+1} \gamma^{m-1} \omega^{p-1} \,, \qquad \hat{\alpha} \equiv \frac{\alpha_s}{2\pi}$$

$$\mathcal{K}_1 = \left(\Gamma_1 - \mathcal{K}_0^{(1)} \mathcal{K}_0^{(0)}\right) \chi_c^{\omega} + (2C_A)^2 \left(\frac{1}{\omega} - \frac{2}{1+\omega}\right) \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\chi}_1^{\omega} - \tilde{\chi}_1^{(0)} \chi_c^{\omega} \end{pmatrix}$$

$$\tilde{\chi}_1^{\omega=0} \equiv \tilde{\chi}_1 = \frac{{}_0\mathcal{K}_{gg,1}}{(2C_A)^2} = K_1^{\mathrm{BFKL}} - \frac{\left[{}_0\mathcal{K}_0 \; {}_1\mathcal{K}_0\right]_{gg}}{(2C_A)^2}$$