NLO evolution of color dipoles

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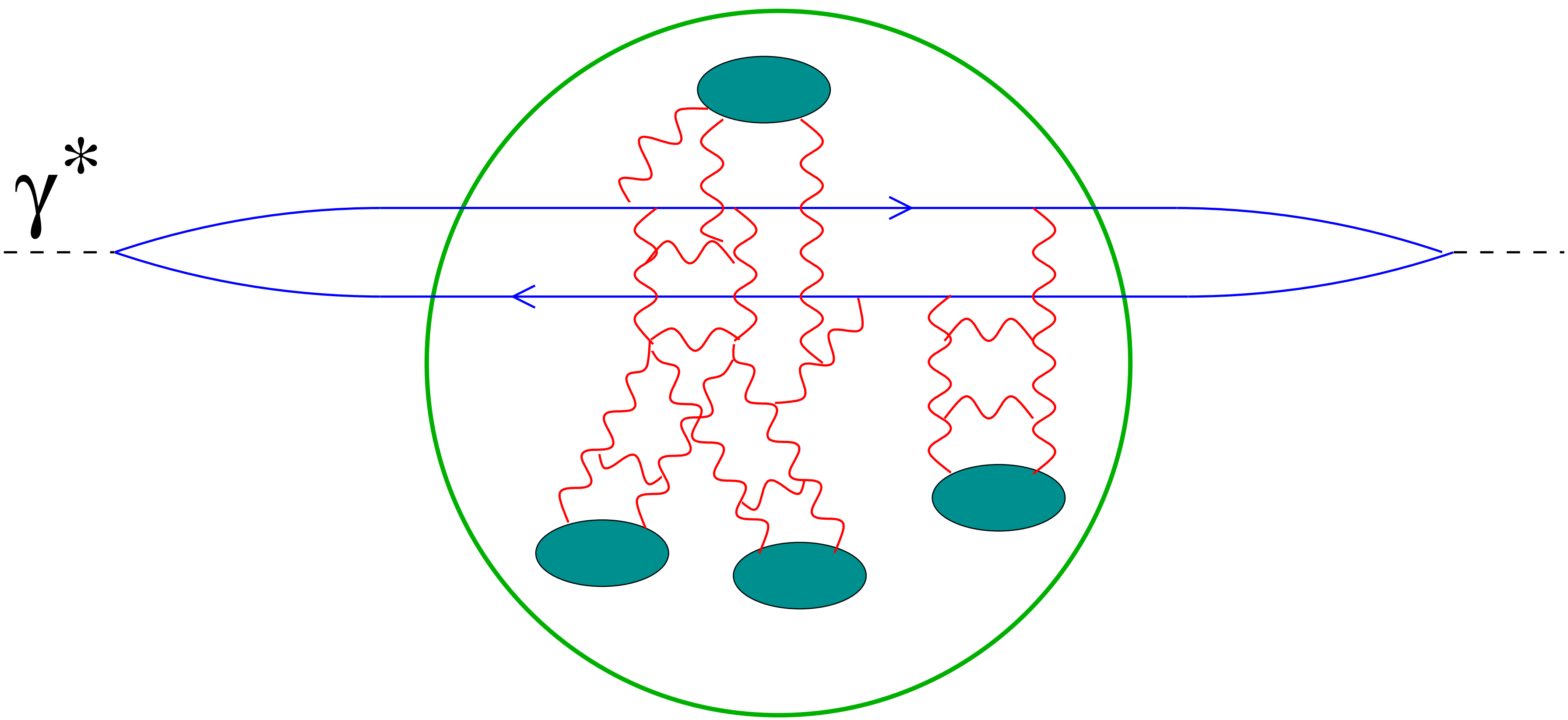
ISMD07

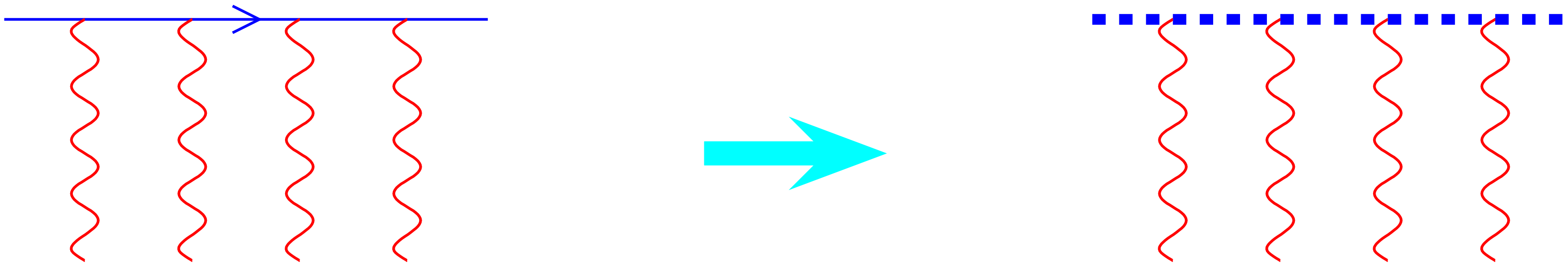
Plan

- Small-x DIS as an evolution of Wilson lines
- Non-linear evolution equation (BK eqn)
- Quark loop contribution to NLO BK.
- Can the high-energy scattering be described in terms of dipoles at the NLO level?
- Bubble chain and the argument of coupling constant
- Gluon part of NLO BK
- Conclusions and outlook

Small-x DIS from the nucleus

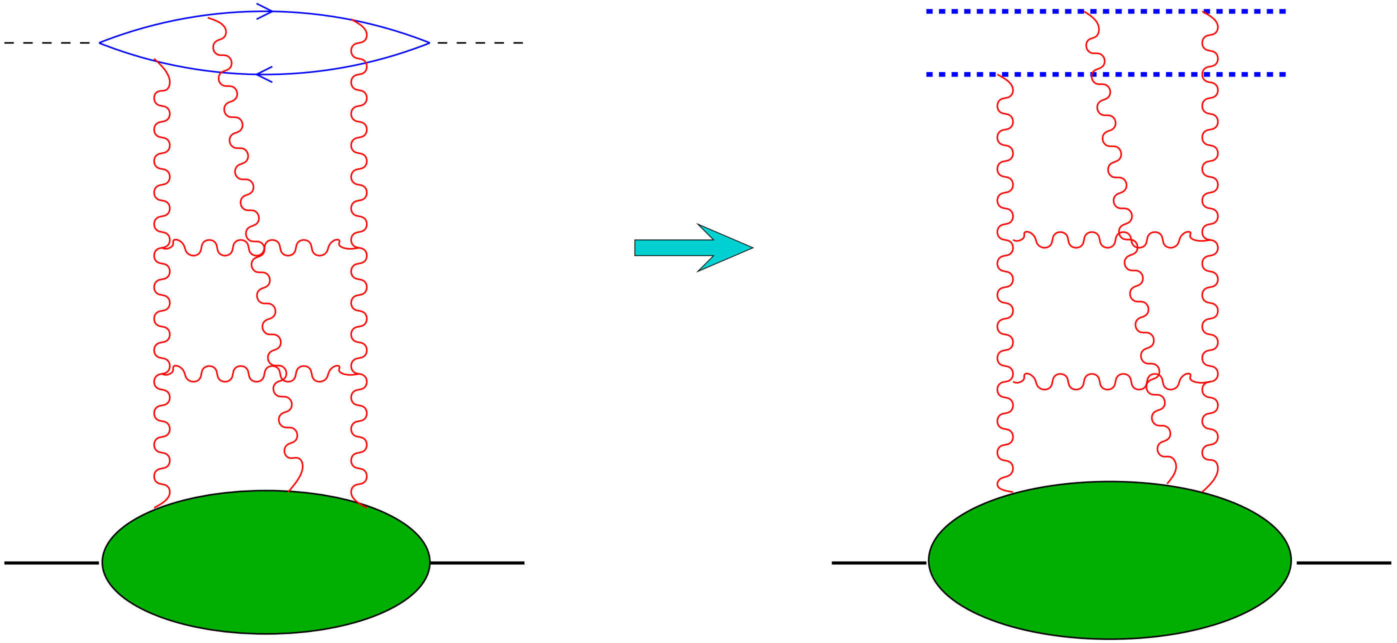


Fast quark moves along the straight line $\Rightarrow$



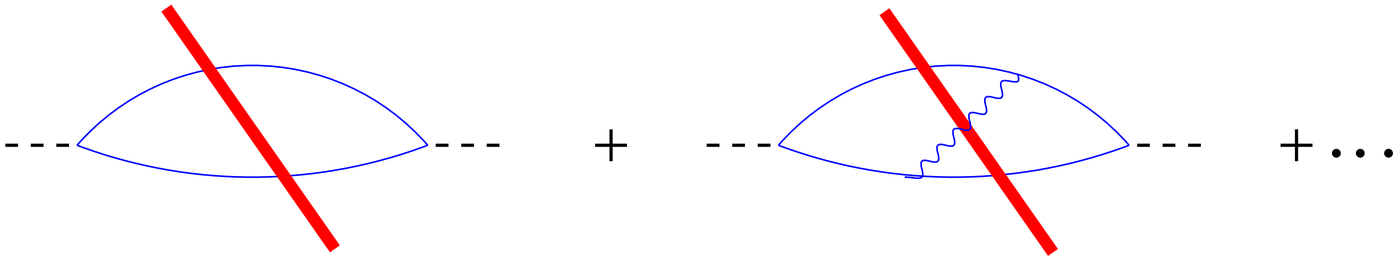
quark propagator reduces to the Wilson line collinear to quark’s velocity

At high energies, the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator ("color dipole").

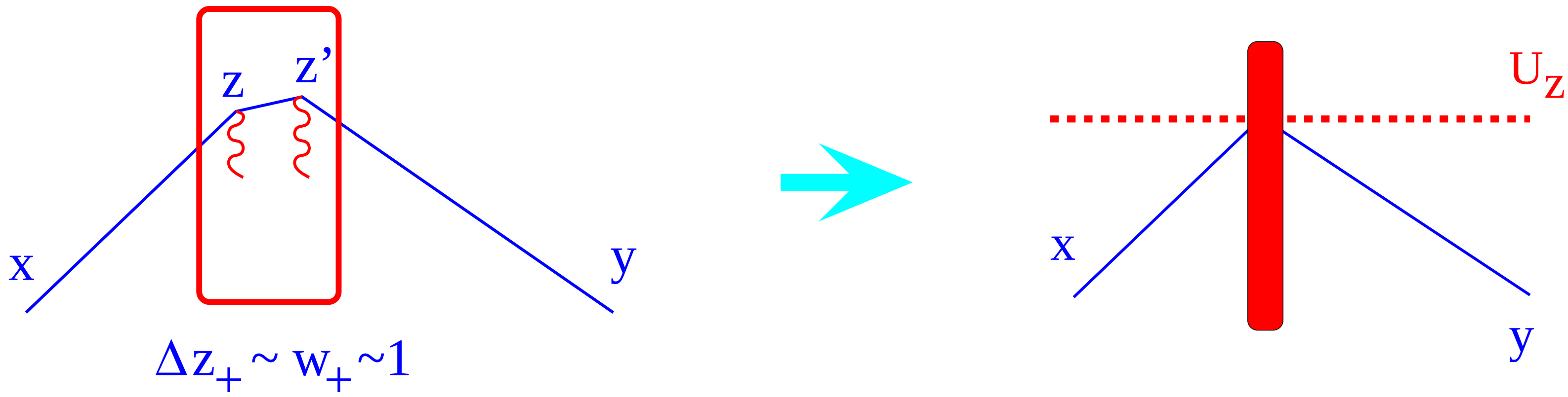


$$A(s) \;=\; \int \frac{d^2k_\perp}{4\pi^2} I(k_\perp)\langle B|\mathrm{Tr}\{U^{\eta_A}(k_\perp)U^{\dagger\eta_A}(-k_\perp)\}|B\rangle + \ldots$$

High-speed nucleus shrinks to a “pancake” $\Rightarrow$



Quarks (and gluons) do not have time to deviate in the transverse direction

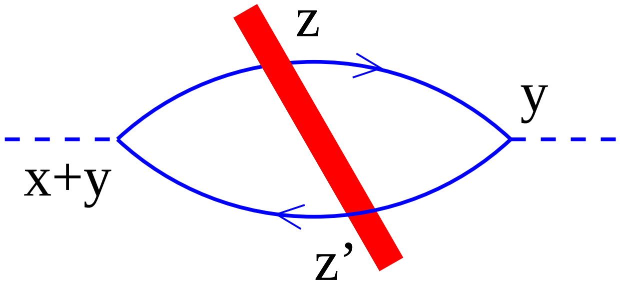


$$|z-z'|_\perp \sim \sqrt{\frac{w_+}{\alpha s}} \sim \sqrt{\frac{1}{s}} \Rightarrow G(x,y) = \int dz\,\delta(z_+)(x|\frac{\not p}{p^2}|z)\ \not p_2 U_z(z|\frac{\not p}{p^2}|y)$$

$$U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp] \quad - \quad \text{Wilson line}$$

$$[x,y] \equiv Pe^{ig\int_0^1 du\,(x-y)^\mu A_\mu(ux+(1-u)y)}$$

Feynman diagrams in a shock-wave background

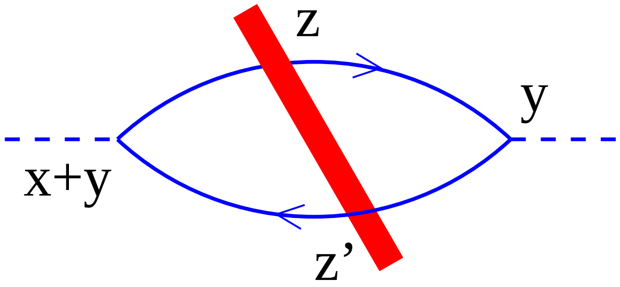


$$\int d^4x d^4y\, e^{-ip_A\cdot x}\langle T\{j_A(x+y)j'_A(y)\}\rangle_A$$

$$= \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp)\mathrm{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\} + \ldots \Rightarrow$$

$$A(s) = \int \frac{d^2k_\perp}{4\pi^2} I(k_\perp)\langle B|\mathrm{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\}|B\rangle +$$

Feynman diagrams in a shock-wave background



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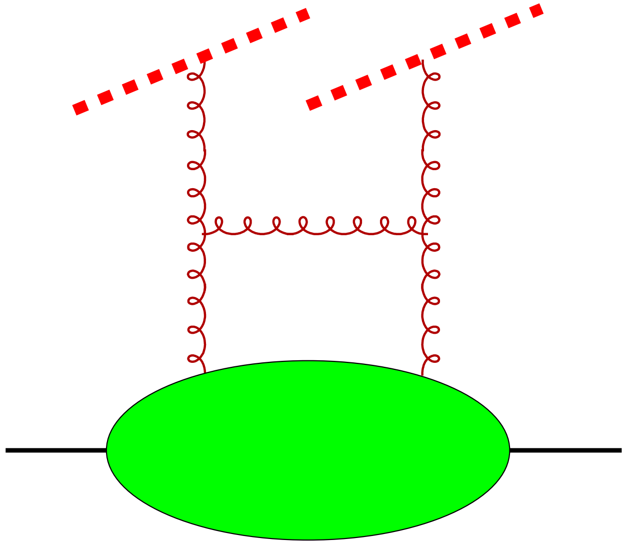
$$= \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp)\mathrm{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\} + \ldots \Rightarrow$$

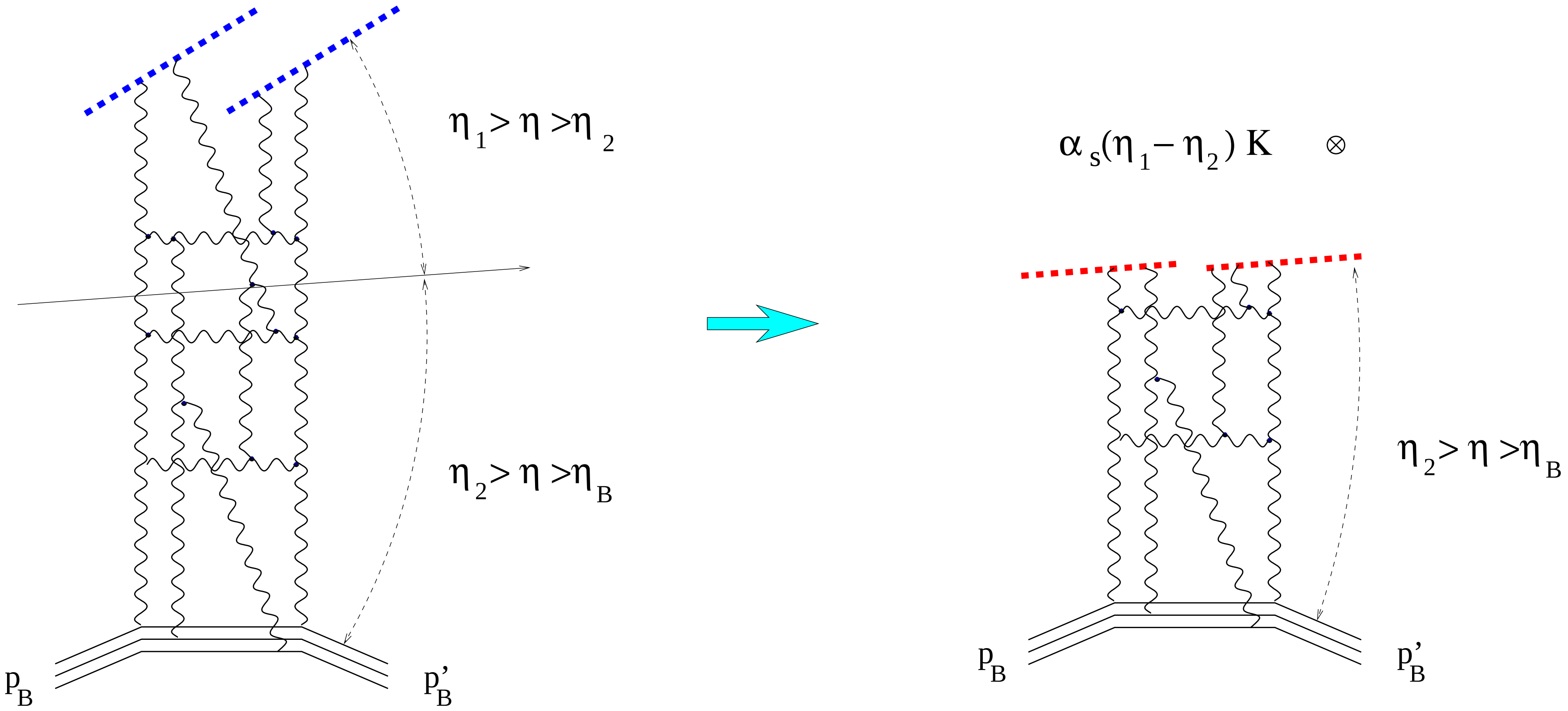
$$A(s) = \int \frac{d^2k_\perp}{4\pi^2} I(k_\perp)\langle B|\mathrm{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\}|B\rangle +$$

Energy dependence of the amplitude $A(s)$ is determined by the dependence of the Wilson lines on the rapidity η_A defined by the slope of the line.

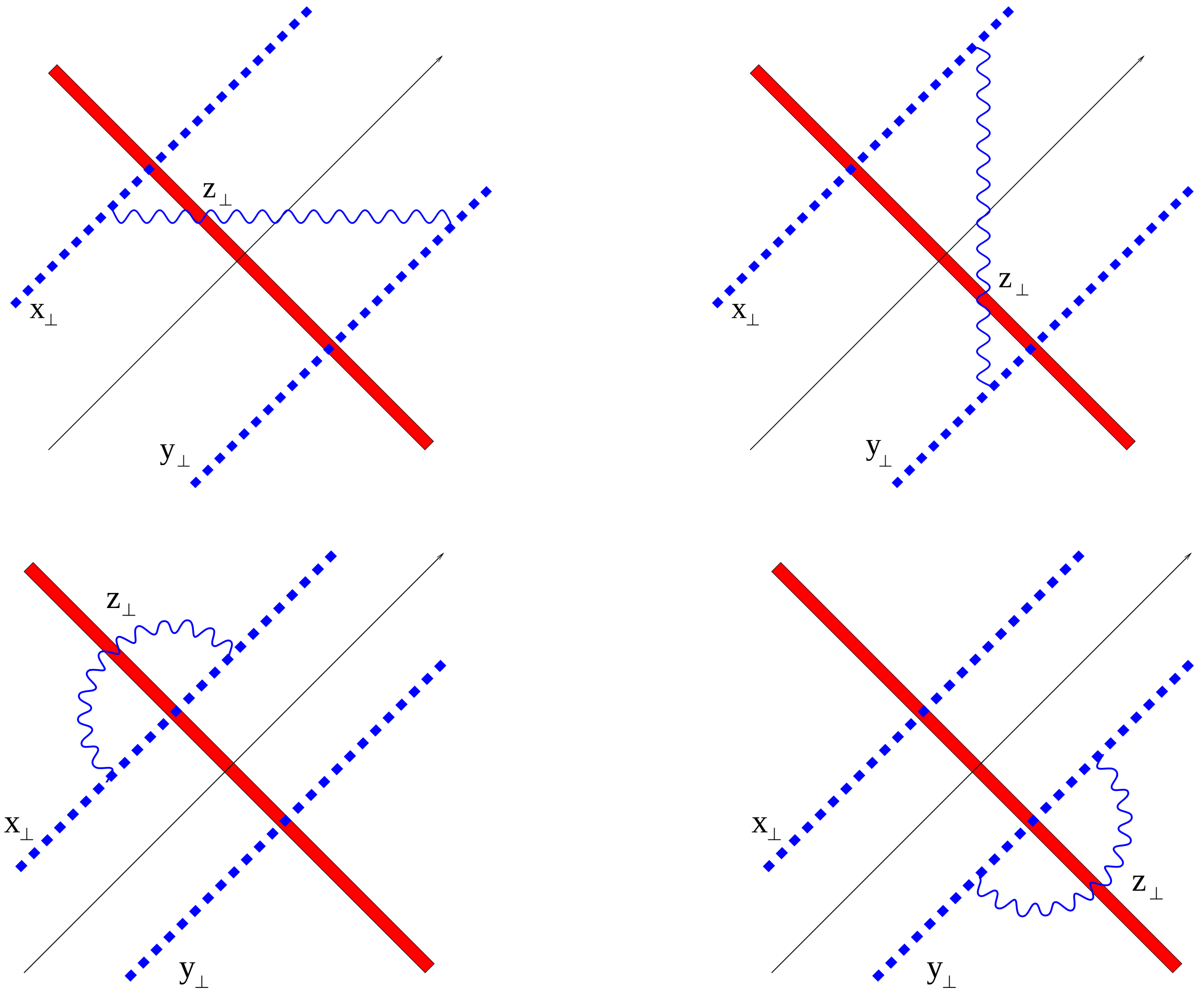


$$\langle B|U_x^{\eta_A}U_y^{\dagger\eta_A}|B\rangle \sim \int_{\eta_B}^{\eta_A} d\eta$$

To get the evolution equation, consider the dipole with the slope $\parallel\ \eta_1$ and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with the slope corresponding to η_2).



In the frame $\parallel\ \eta_1$ the gluons with $\eta < \eta_2$ are seen as a pancake $\Rightarrow$



The structure is

[$x \longrightarrow z$: free propagation]

$\times$

[$U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave]

$\times$

[$z \longrightarrow y$: free propagation]

$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \;\Rightarrow (U_x U_y^\dagger)^{\eta_1} = (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$\Rightarrow$ non-linear evolution

Non-linear evolution equation

$$\frac{\partial}{\partial\eta}\mathcal{U}(x_\perp, y_\perp) \ =$$

$$-\frac{\bar{\alpha}}{4\pi}\int dz_\perp \frac{(\vec{x}-\vec{y})^2_\perp}{(\vec{x}_\perp - \vec{z}_\perp)^2(\vec{z}_\perp - \vec{y}_\perp)^2}$$

$$\times \left\{\mathcal{U}(x_\perp, z_\perp) + \mathcal{U}(z_\perp, y_\perp) - \mathcal{U}(x_\perp, y_\perp) - \mathcal{U}(x_\perp, z_\perp)\mathcal{U}(z_\perp, y_\perp)\right\}$$

$$\mathcal{U}(x_\perp, y_\perp) \equiv \frac{1}{N_c}(N_c - \mathrm{Tr}\{U(x_\perp)U^\dagger(y_\perp)\})$$

LLA for DIS in pQCD $\Rightarrow$ BFKL
LLA for DIS in sQCD $\Rightarrow$ BK eqn
(s for semiclassical)

$$\frac{\partial}{\partial\eta}\mathcal{U}(x_\perp, y_\perp) \; =$$

$$-\frac{\bar{\alpha}}{4\pi}\int dz_\perp \frac{(\vec{x}-\vec{y})^2_\perp}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{z}_\perp-\vec{y}_\perp)^2}$$

$$\times\left\{\underbrace{\mathcal{U}(x_\perp, z_\perp)+\mathcal{U}(z_\perp, y_\perp)-\mathcal{U}(x_\perp, y_\perp)}-\mathcal{U}(x_\perp, z_\perp)\mathcal{U}(z_\perp, y_\perp)\right\}$$

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$$\times \left\{\mathcal{U}(x_\perp, z_\perp) + \mathcal{U}(z_\perp, y_\perp) - \mathcal{U}(x_\perp, y_\perp) - \underbrace{\mathcal{U}(x_\perp, z_\perp)\mathcal{U}(z_\perp, y_\perp)}\right\}$$

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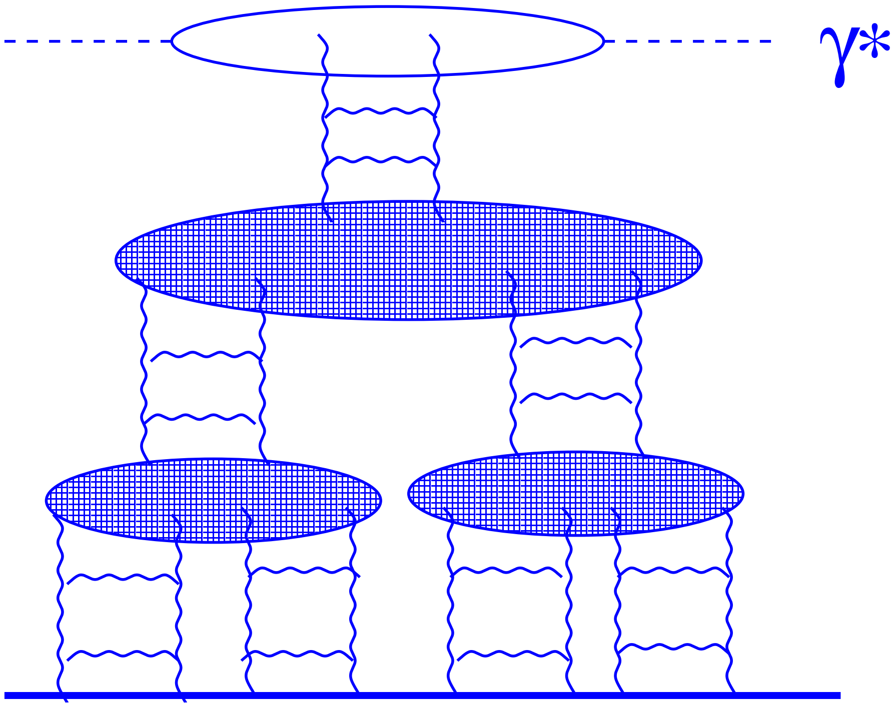
LLA for DIS in pQCD $\Rightarrow$ BFKL

LLA for DIS in sQCD $\Rightarrow$ BK eqn

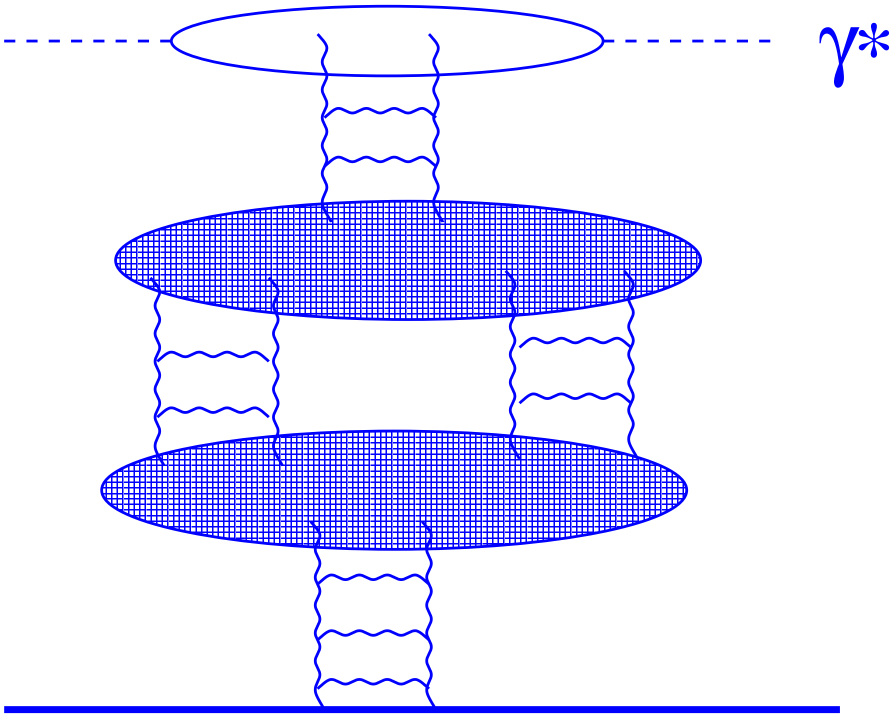
(s for semiclassical)

Example - LLA for the structure functions of large nuclei: $\alpha_s \ln\frac{1}{x} \sim 1$, $\alpha_s^2 A^{1/3} \sim 1$

Non-linear equation sums up the "fan" diagrams



Example of the diagrams left behind by the NL eqn: pomeron loops



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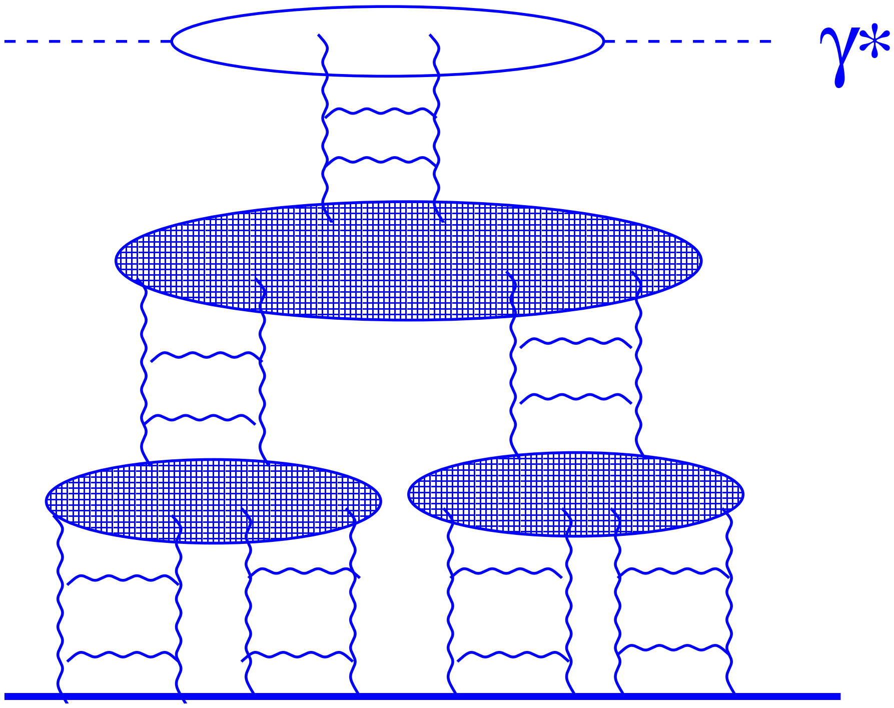

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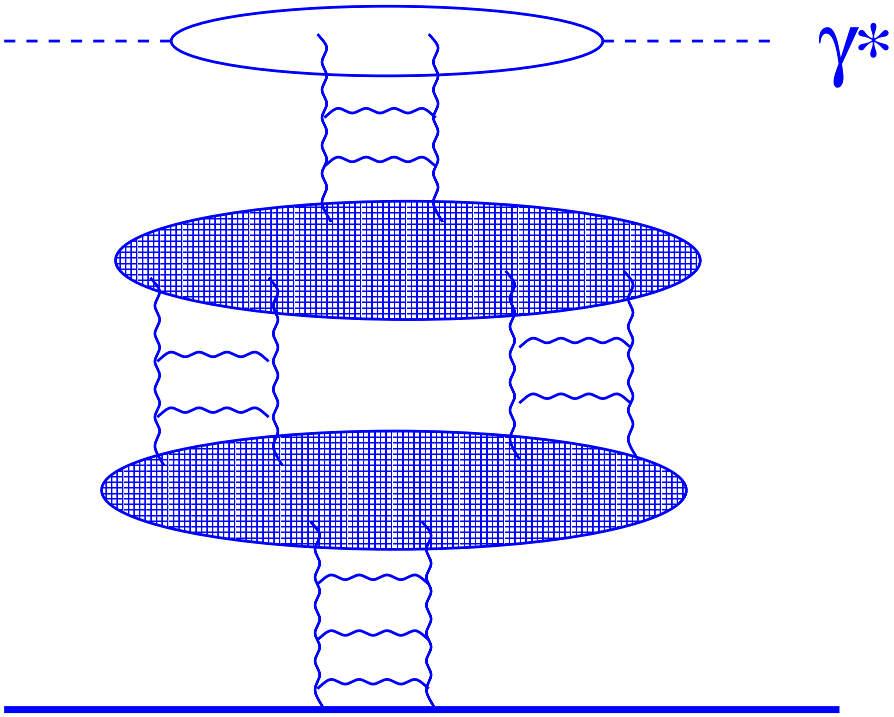
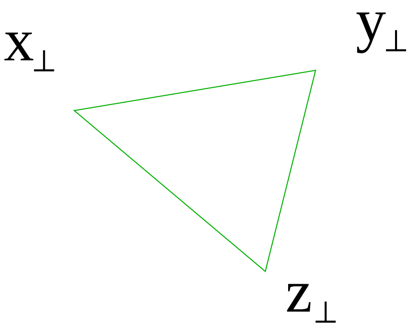


$x_B \to 0 \overset{\text{BFKL}}{\Rightarrow}$ gluon density increases $\overset{\text{BK}}{\Rightarrow}$ saturation $\Rightarrow$ CGC

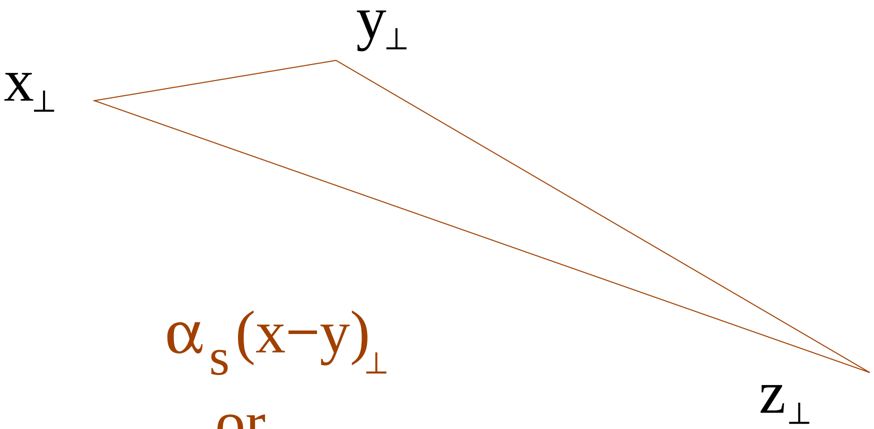
Argument of the α_s in the BK equation

$$\frac{\partial}{\partial\eta}\mathcal{U}(x_\perp,y_\perp) \;=\; \frac{\alpha_s(?_\perp)}{2\pi^2}\int dz_\perp \frac{(\vec{x}-\vec{y})^2_\perp}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{z}_\perp-\vec{y}_\perp)^2}$$
$$\times\left[\mathcal{U}(x_\perp,z_\perp)+\mathcal{U}(z_\perp,y_\perp)-\mathcal{U}(x_\perp,y_\perp)-\mathcal{U}(x_\perp,z_\perp)\mathcal{U}(z_\perp,y_\perp)\right]$$



$(x-y)_\perp \sim (y-z)_\perp \sim (x-z)_\perp$

→ α_s(whatever)



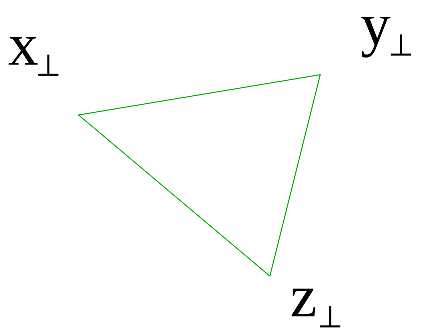
$\alpha_s(x-y)_\perp$

or

$\alpha_s(x-z)_\perp$?

Argument of the α_s in the BK equation

$$\frac{\partial}{\partial \eta}\mathcal{U}(x_\perp, y_\perp) = \frac{\alpha_s(?_\perp)}{2\pi^2}\int dz_\perp \frac{(\vec{x}-\vec{y})^2_\perp}{(\vec{x}_\perp - \vec{z}_\perp)^2(\vec{z}_\perp - \vec{y}_\perp)^2}$$
$$\times \left[\mathcal{U}(x_\perp, z_\perp) + \mathcal{U}(z_\perp, y_\perp) - \mathcal{U}(x_\perp, y_\perp) - \mathcal{U}(x_\perp, z_\perp)\mathcal{U}(z_\perp, y_\perp)\right]$$



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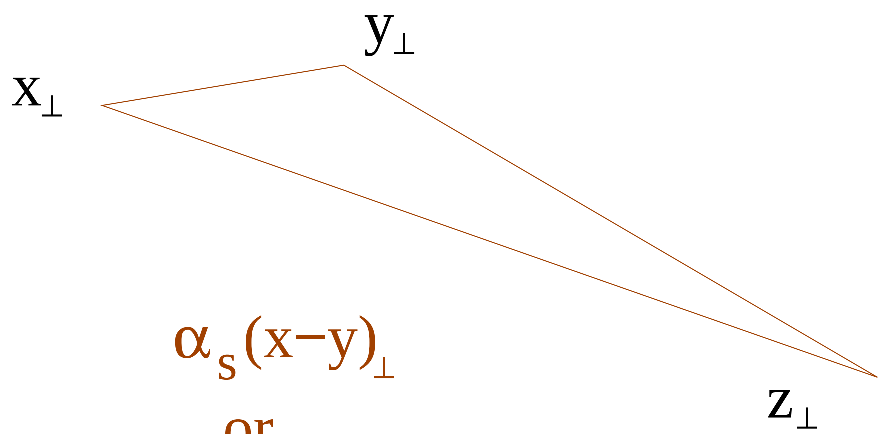
→ α_s(whatever)



$\alpha_s(x-y)_\perp$

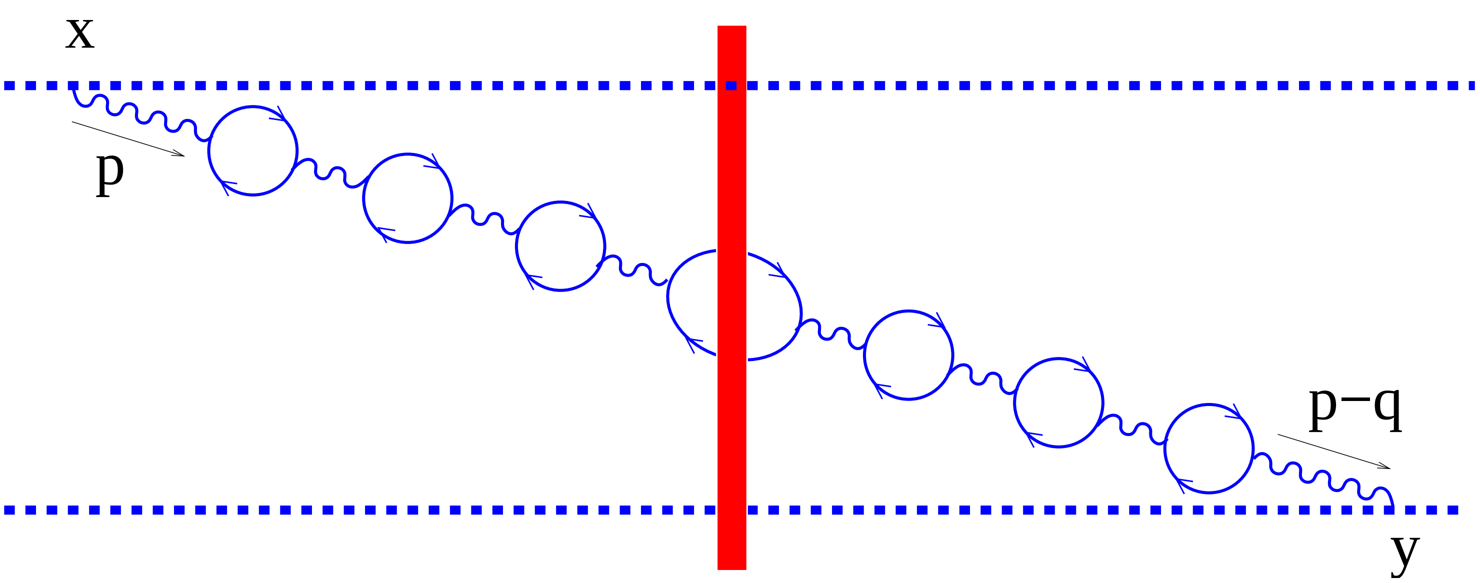
or

$\alpha_s(x-z)_\perp$?

Result: $\alpha_s = \alpha_s(\min\{|x-y|, |x-z|, |y-z|\}_\perp)$

Quark bubble chain and the argument of α_s

$$\alpha_s(p_\perp^2) = \frac{\alpha_s(\mu^2)}{1 + \left(\frac{11}{3}N_c - \frac{2}{3}n_f\right)\frac{\alpha_s}{4\pi}\ln\frac{p_\perp^2}{\mu^2}}$$



Quark bubble chain and the argument of α_s

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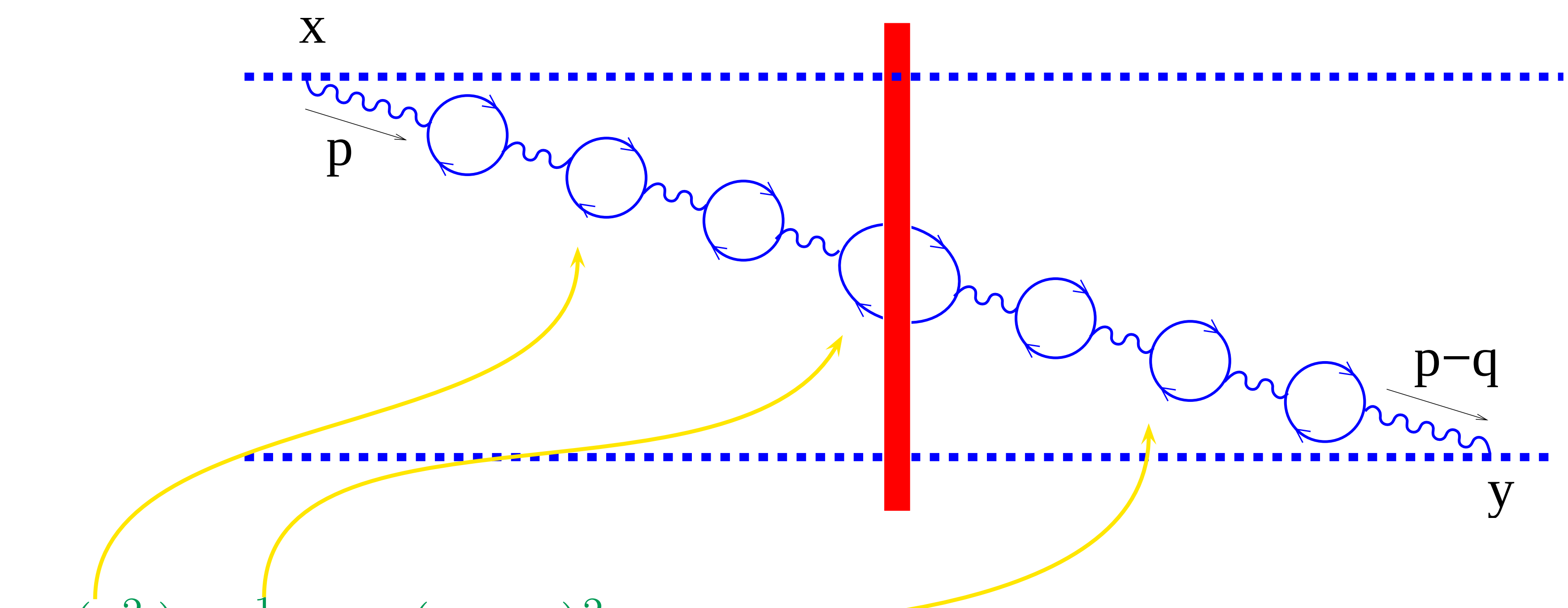
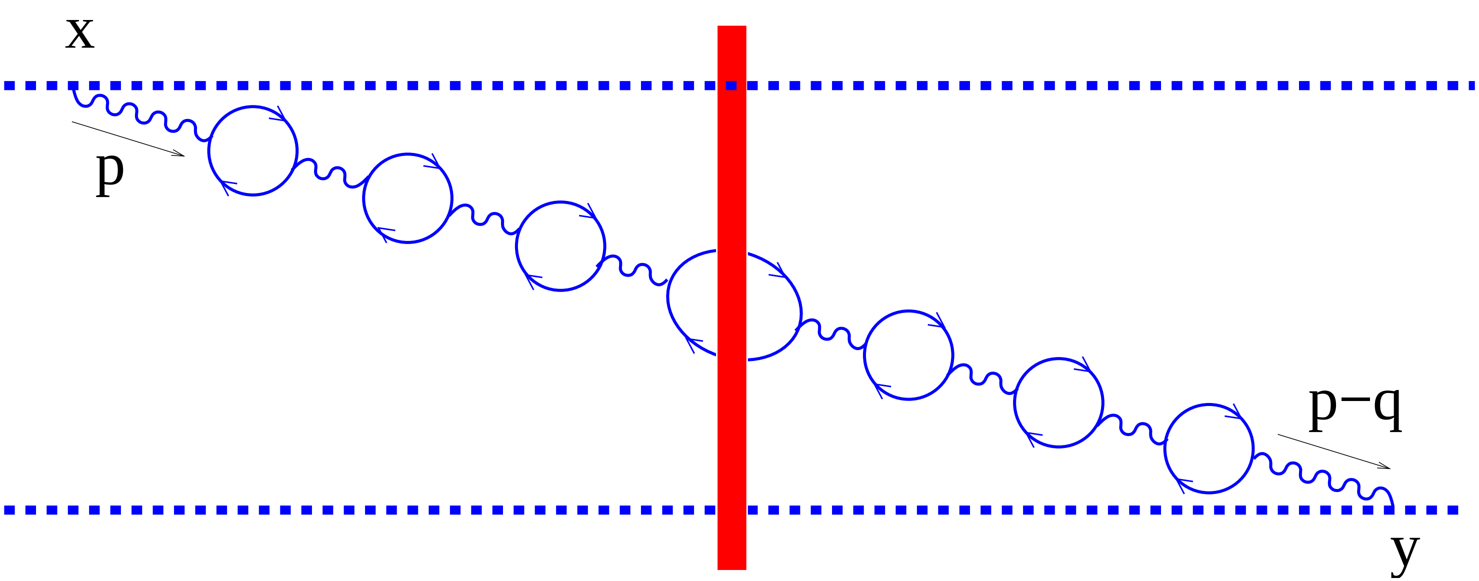


$$\alpha_s(p_\perp^2)\ \frac{1}{\alpha_s(q_\perp^2)}\ \alpha_s(p-q)_\perp^2$$

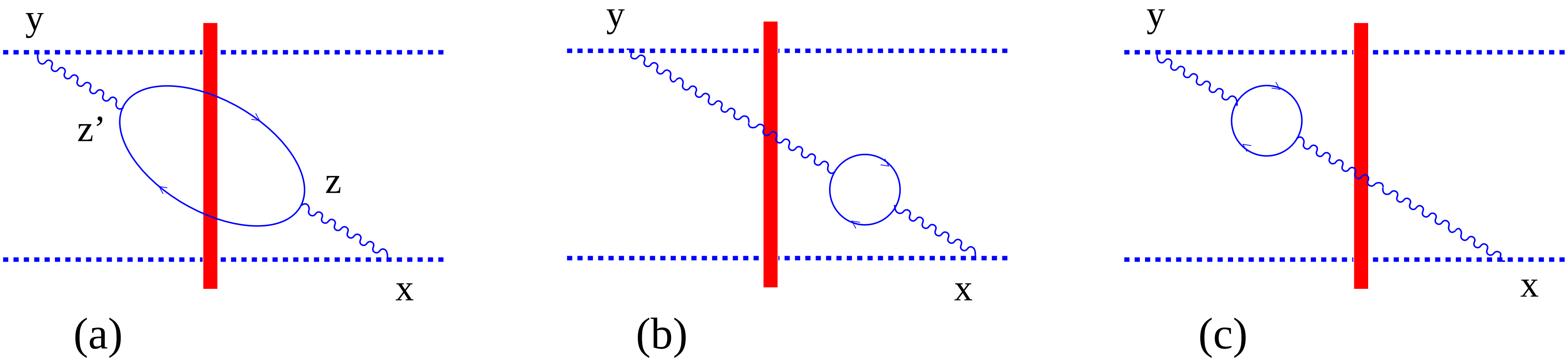
Quark bubble chain and the argument of α_s

$$\alpha_s(p_\perp^2) = \frac{\alpha_s(\mu^2)}{1 + \left(\frac{11}{3}N_c - \frac{2}{3}n_f\right)\frac{\alpha_s}{4\pi}\ln\frac{p_\perp^2}{\mu^2}}$$

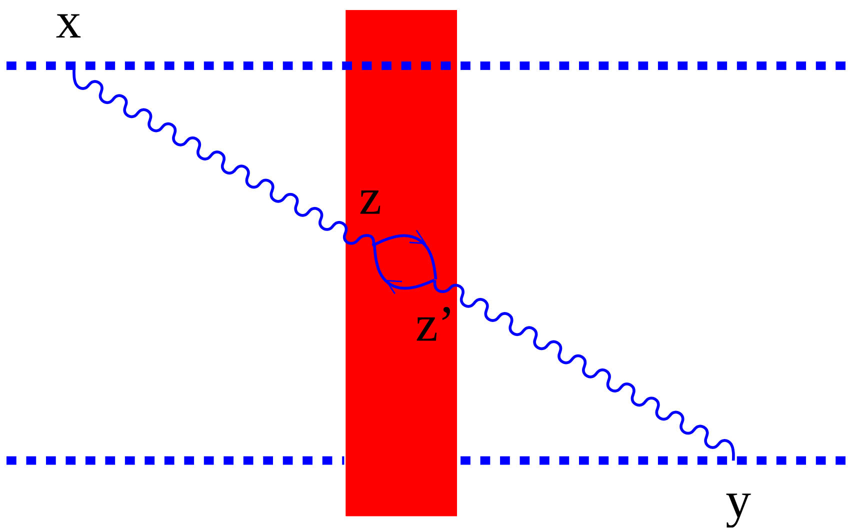


$\alpha_s(p_\perp^2)\,\frac{1}{\alpha_s(q_\perp^2)}\,\alpha_s(p-q)_\perp^2 \quad \Rightarrow \quad \alpha_s(x_i - x_j)_{\min}$ after the Fourier transformation

Quark contribution to the NLO kernel



A problem: quark loop inside the shock wave

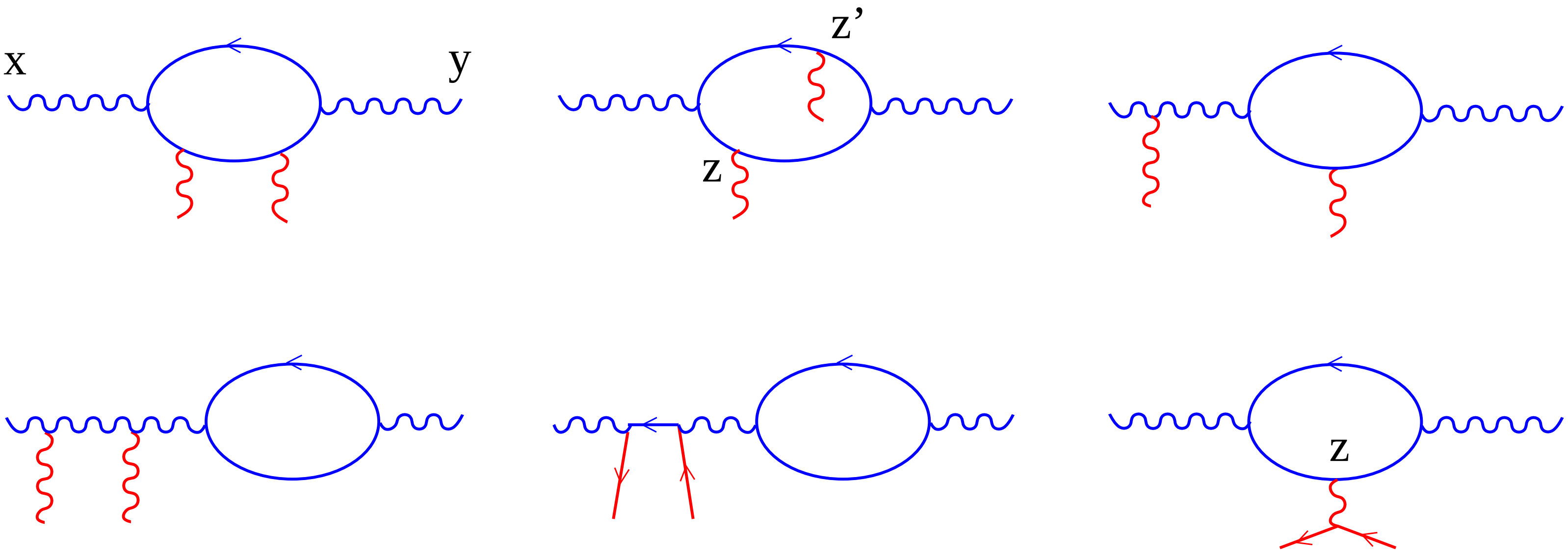


$|z - z'|_\perp^2 \sim \frac{1}{\alpha s} \Rightarrow$ one can expand the quark loop near the light cone $\Rightarrow$ the contribution is local in $z_\perp$.

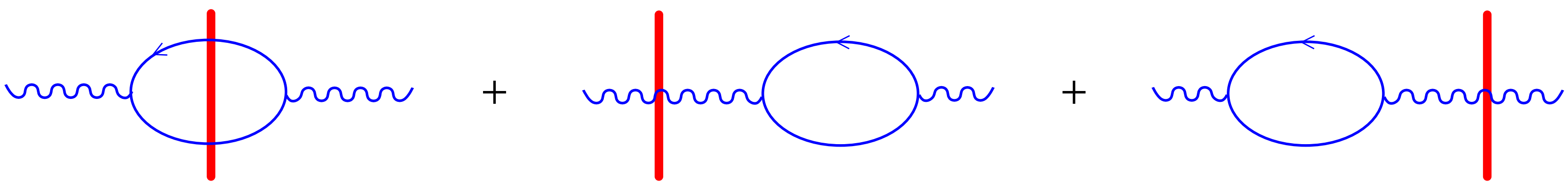
A way to fish out such extra local term is to find the light-cone expansion of $U_x U_y^\dagger$ as $x_\perp \rightarrow y_\perp$ (up to twist-4 terms) and compare it to our result in the shock-wave background.

Result of the light-cone expansion:

The light-cone expansion of the sum of the diagrams



coincides with the expansion of



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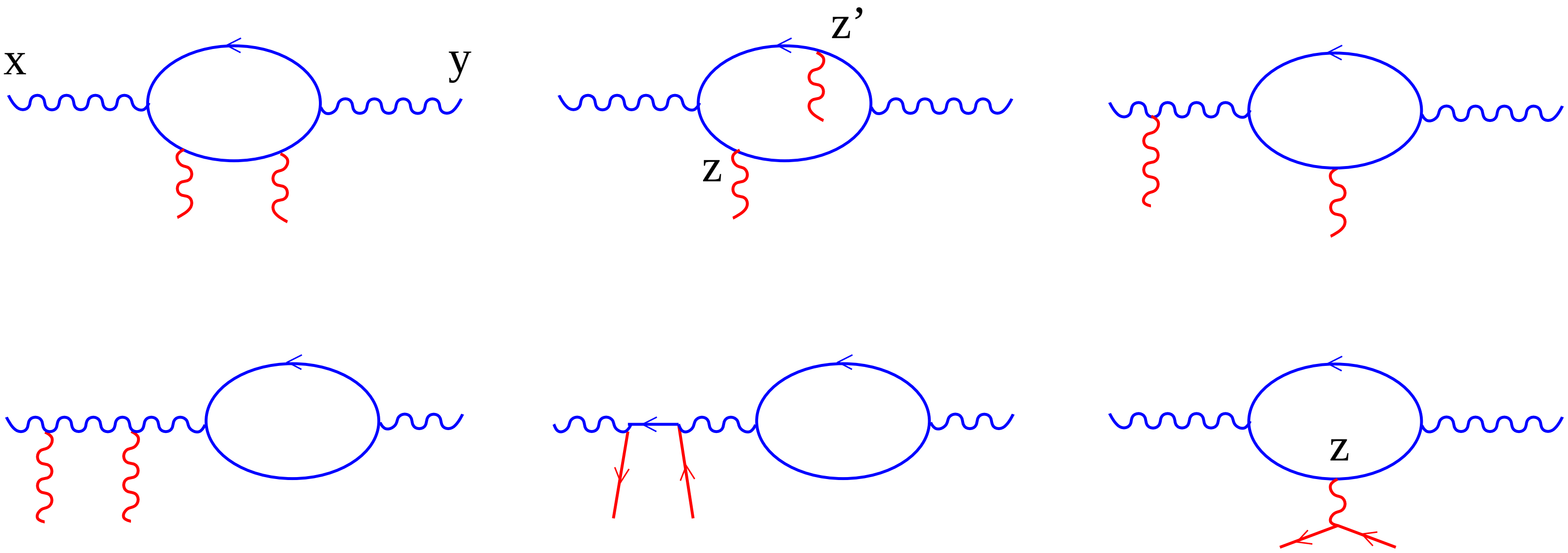

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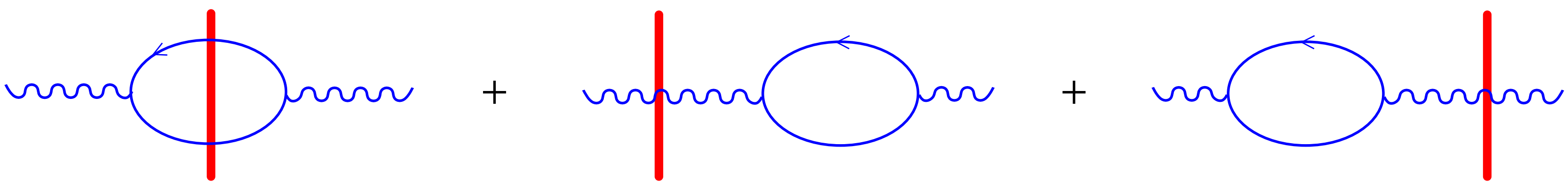
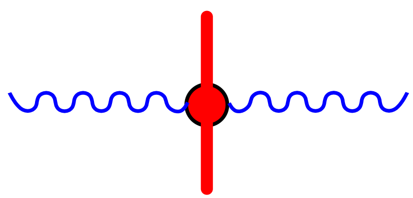
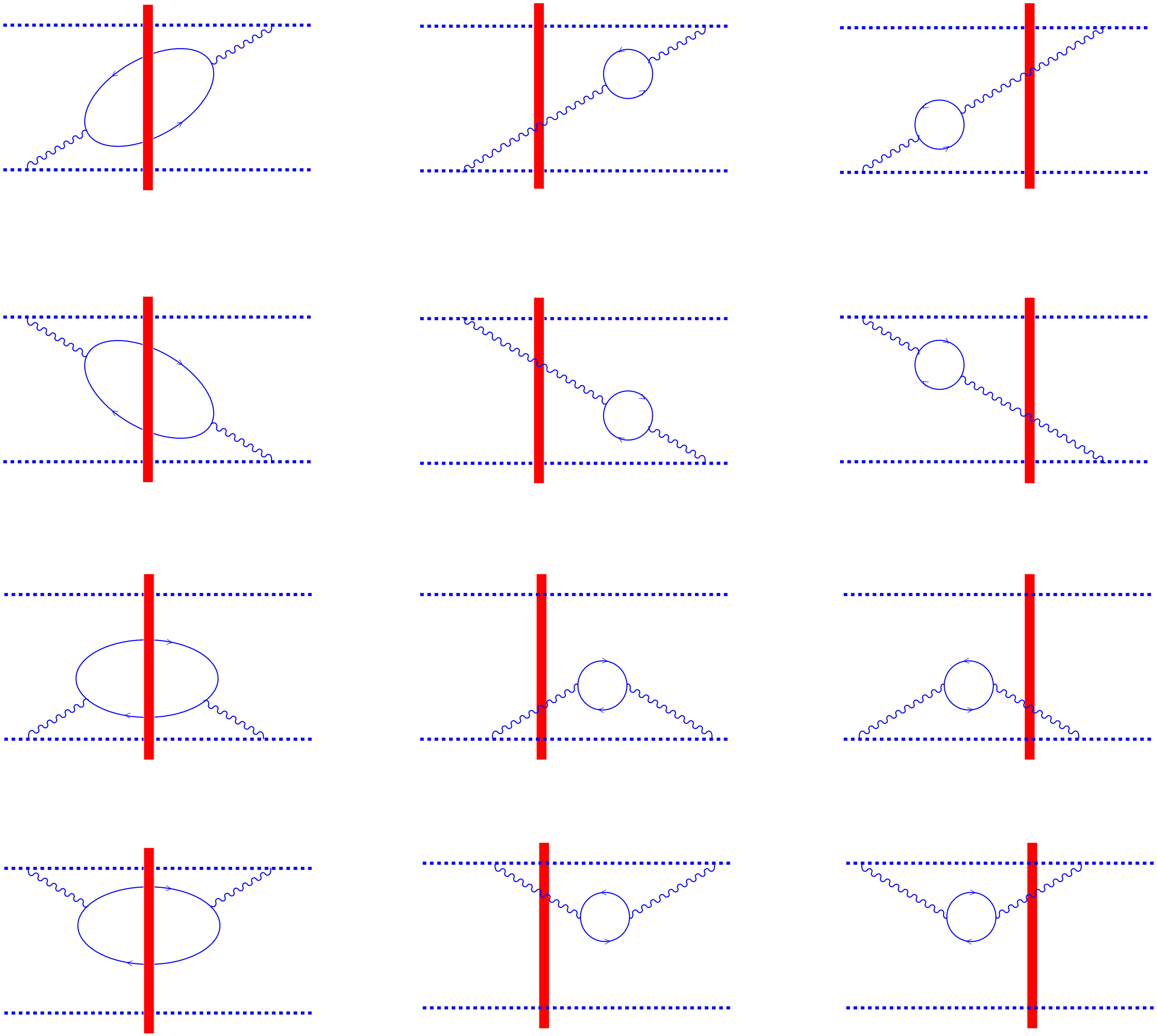


$\Rightarrow$ no additional vertex  at the one−loop level

Diagrams for the dipole evolution



Quark-loop contribution to NLO BK

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}]$$

$$\times\Big[\frac{(x-y)^2}{X^2Y^2}\Big(1-\frac{\alpha_s n_f}{6\pi}[\ln(\mathbf{x}-\mathbf{y})^2\mu^2+\frac{5}{3}]\Big)+\frac{\alpha_s n_f}{6\pi}\frac{\mathbf{X^2-Y^2}}{\mathbf{X^2Y^2}}\ln\frac{\mathbf{X^2}}{\mathbf{Y^2}}\Big]$$

$$+[\frac{\alpha_s^2}{\pi^4}n_f\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}\int d^2zd^2z'\,\mathrm{Tr}\{t^aU_zt^bU_{z'}^\dagger - t^aU_zt^bU_z^\dagger\}\frac{1}{(z-z')^4}$$

$$\times\Big\{1-\frac{\mathbf{X'^2Y^2+Y'^2X^2-(x-y)^2(z-z')^2}}{\mathbf{2(X'^2Y^2-Y'^2X^2)}}\ln\frac{\mathbf{X'^2Y^2}}{\mathbf{Y'^2X^2}}\Big\}$$

$$X=x-z,\;\; X'=x-z',\;\; Y=y-z,\;\; Y'=y-z'$$

Quark-loop contribution to NLO BK

$$\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\, [\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c \mathrm{Tr}\{U_x U_y^\dagger\}]$$

$$\times \Big[\frac{(x-y)^2}{X^2Y^2}\Big(1 - \frac{\alpha_s n_f}{6\pi}[\ln(\mathbf{x}-\mathbf{y})^2\mu^2 + \tfrac{5}{3}]\Big) + \frac{\alpha_s n_f}{6\pi}\frac{\mathbf{X^2}-\mathbf{Y^2}}{\mathbf{X^2Y^2}}\ln\frac{\mathbf{X^2}}{\mathbf{Y^2}}\Big]$$

$$+[\frac{\alpha_s^2}{\pi^4}n_f \mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}\int d^2z d^2z'\, \mathrm{Tr}\{t^aU_zt^bU_{z'}^\dagger - t^aU_zt^bU_z^\dagger\}\frac{1}{(z-z')^4}$$

$$\times \Big\{1 - \frac{\mathbf{X'^2Y^2}+\mathbf{Y'^2X^2}-(\mathbf{x}-\mathbf{y})^2(\mathbf{z}-\mathbf{z'})^2}{2(\mathbf{X'^2Y^2}-\mathbf{Y'^2X^2})}\ln\frac{\mathbf{X'^2Y^2}}{\mathbf{Y'^2X^2}}\Big\}$$

$$X = x - z,\ \ X' = x - z',\ \ Y = y - z,\ \ Y' = y - z'$$

Running coupling part

Quark-loop contribution to NLO BK

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}]$$

$$\times\Big[\frac{(x-y)^2}{X^2Y^2}\Big(1-\frac{\alpha_s n_f}{6\pi}[\ln(\mathbf{x}-\mathbf{y})^2\mu^2+\tfrac{5}{3}]\Big)+\frac{\alpha_s n_f}{6\pi}\frac{\mathbf{X}^2-\mathbf{Y}^2}{\mathbf{X}^2\mathbf{Y}^2}\ln\frac{\mathbf{X}^2}{\mathbf{Y}^2}\Big]$$

$$+[\frac{\alpha_s^2}{\pi^4}n_f\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}\int d^2z d^2z'\,\mathrm{Tr}\{t^aU_zt^bU_{z'}^\dagger - t^aU_zt^bU_z^\dagger\}\frac{1}{(z-z')^4}$$

$$\times\Big\{1-\frac{\mathbf{X}'^2\mathbf{Y}^2+\mathbf{Y}'^2\mathbf{X}^2-(\mathbf{x}-\mathbf{y})^2(\mathbf{z}-\mathbf{z}')^2}{2(\mathbf{X}'^2\mathbf{Y}^2-\mathbf{Y}'^2\mathbf{X}^2)}\ln\frac{\mathbf{X}'^2\mathbf{Y}^2}{\mathbf{Y}'^2\mathbf{X}^2}\Big\}$$

$$X=x-z,\;\; X'=x-z',\;\; Y=y-z,\;\; Y'=y-z'$$

Running coupling part $\quad + \quad$ Conformal part

Quark-loop contribution to NLO BK

$$\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}]$$
$$\times\Big[\frac{(x-y)^2}{X^2Y^2}\Big(1-\frac{\alpha_s n_f}{6\pi}[\ln(\mathbf{x}-\mathbf{y})^2\mu^2+\tfrac{5}{3}]\Big)+\frac{\alpha_s n_f}{6\pi}\frac{\mathbf{X^2-Y^2}}{\mathbf{X^2Y^2}}\ln\frac{\mathbf{X^2}}{\mathbf{Y^2}}\Big]$$
$$+[\frac{\alpha_s^2}{\pi^4}n_f\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}\int d^2z d^2z'\;\mathrm{Tr}\{t^aU_zt^bU_{z'}^\dagger - t^aU_zt^bU_z^\dagger\}\frac{1}{(z-z')^4}$$
$$\times\Big\{1-\frac{\mathbf{X'^2Y^2+Y'^2X^2-(x-y)^2(z-z')^2}}{\mathbf{2(X'^2Y^2-Y'^2X^2)}}\ln\frac{\mathbf{X'^2Y^2}}{\mathbf{Y'^2X^2}}\Big\}$$

$$X=x-z,\;\; X'=x-z',\;\; Y=y-z,\;\; Y'=y-z'$$

$$\text{Running coupling} = \alpha_s(\mu)\Big\{1-\frac{\alpha_s n_f}{6\pi}\Big[\ln(x-y)^2\mu^2-\frac{X^2-Y^2}{(x-y)^2}\ln\frac{X^2}{Y^2}\Big]$$

Quark-loop contribution to NLO BK

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}]$$

$$\times\Big[\frac{(x-y)^2}{X^2Y^2}\Big(1-\frac{\alpha_s n_f}{6\pi}[\ln(\mathbf{x}-\mathbf{y})^2\mu^2+\tfrac{5}{3}]\Big)+\frac{\alpha_s n_f}{6\pi}\frac{\mathbf{X^2-Y^2}}{\mathbf{X^2Y^2}}\ln\frac{\mathbf{X^2}}{\mathbf{Y^2}}\Big]$$

$$+[\frac{\alpha_s^2}{\pi^4}n_f\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}\int d^2zd^2z'\,\mathrm{Tr}\{t^aU_zt^bU_{z'}^\dagger - t^aU_zt^bU_z^\dagger\}\frac{1}{(z-z')^4}$$

$$\times\Big\{1-\frac{\mathbf{X'^2Y^2+Y'^2X^2-(x-y)^2(z-z')^2}}{\mathbf{2(X'^2Y^2-Y'^2X^2)}}\ln\frac{\mathbf{X'^2Y^2}}{\mathbf{Y'^2X^2}}\Big\}$$

$$X=x-z,\;\; X'=x-z',\;\; Y=y-z,\;\; Y'=y-z'$$

$$\text{Running coupling} = \alpha_s(\mu)\Big\{1-\frac{\alpha_s n_f}{6\pi}\Big[\ln(x-y)^2\mu^2-\frac{X^2-Y^2}{(x-y)^2}\ln\frac{X^2}{Y^2}\Big]$$

$$+\ \text{gluon loop}$$

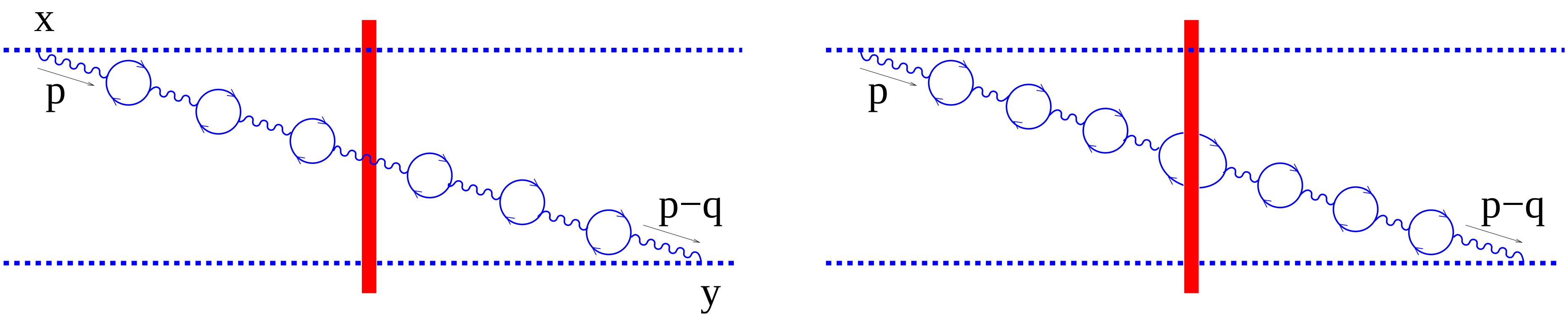
Quark-loop contribution to NLO BK

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\, [\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}]$$
$$\times\Big[\frac{(x-y)^2}{X^2Y^2}\Big(1-\frac{\alpha_s n_f}{6\pi}[\ln(\mathbf{x}-\mathbf{y})^2\mu^2+\tfrac{5}{3}]\Big)+\frac{\alpha_s n_f}{6\pi}\frac{\mathbf{X^2-Y^2}}{\mathbf{X^2Y^2}}\ln\frac{\mathbf{X^2}}{\mathbf{Y^2}}\Big]$$
$$+[\frac{\alpha_s^2}{\pi^4}n_f\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}\int d^2zd^2z'\ \mathrm{Tr}\{t^aU_zt^bU_{z'}^\dagger - t^aU_zt^bU_z^\dagger\}\frac{1}{(z-z')^4}$$
$$\times\Big\{1-\frac{\mathbf{X'^2Y^2+Y'^2X^2-(x-y)^2(z-z')^2}}{\mathbf{2(X'^2Y^2-Y'^2X^2)}}\ln\frac{\mathbf{X'^2Y^2}}{\mathbf{Y'^2X^2}}\Big\}$$

$$X=x-z,\ \ X'=x-z',\ \ Y=y-z,\ \ Y'=y-z'$$

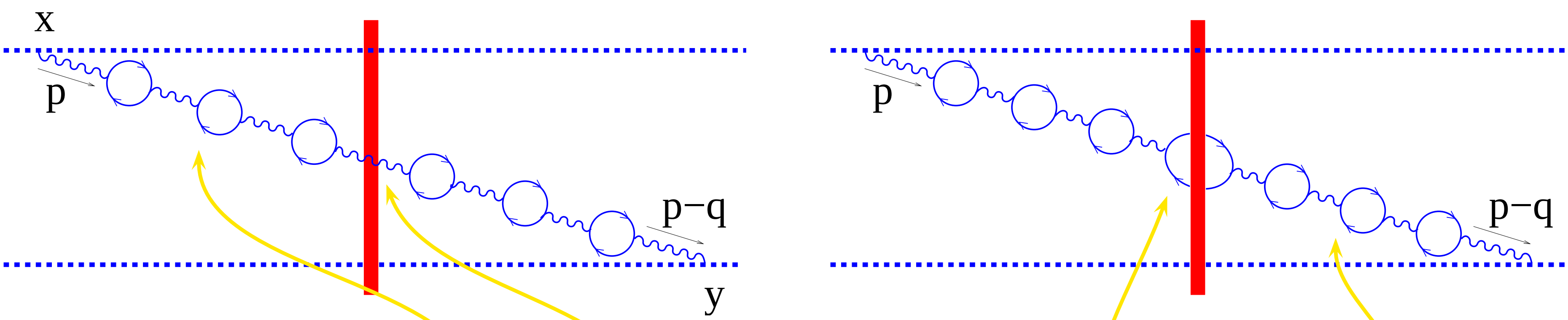
$$\text{Running coupling} = \alpha_s(\mu)\Big\{1-\frac{\alpha_s n_f}{6\pi}\Big[\ln(x-y)^2\mu^2-\frac{X^2-Y^2}{(x-y)^2}\ln\frac{X^2}{Y^2}\Big]$$
$$+\ \text{gluon loop}\ \simeq\ \alpha_s(|x_i-x_j|_{\min})$$

Bubble chain and the argument of coupling constant



Leading log accuracy: $\alpha(\mu) \ll 1$, $\alpha(\mu) \ln \frac{p_\perp^2}{\mu^2} \sim 1$

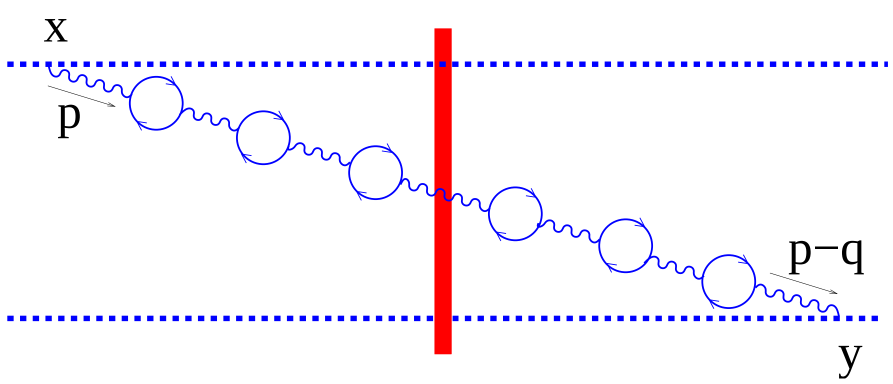
Bubble chain and the argument of coupling constant

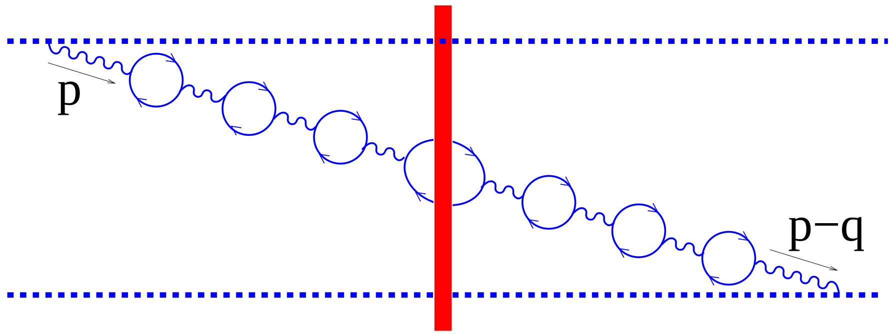


$$LLA \Rightarrow \frac{\alpha_s(p_\perp^2)}{p_\perp^2}\left[\frac{1}{\alpha_s(\mu^2)} + \left(\frac{11}{3}N_c - \frac{2}{3}n_f\right)\ln\frac{q_\perp^2}{\mu^2}\right]\frac{\alpha_s(p-q)_\perp^2}{(p-q)_\perp^2}$$

$$= \frac{\alpha_s(p_\perp^2)}{p_\perp^2}\,\frac{1}{\alpha_s(q_\perp^2)}\,\frac{\alpha_s(p-q)_\perp^2}{(p-q)_\perp^2}$$

Bubble chain and the argument of coupling constant





$$LLA \Rightarrow \frac{\alpha_s(p_\perp^2)}{p_\perp^2}\left[\frac{1}{\alpha_s(\mu^2)} + \left(\frac{11}{3}N_c - \frac{2}{3}n_f\right)\ln\frac{q_\perp^2}{\mu^2}\right]\frac{\alpha_s(p-q)_\perp^2}{(p-q)_\perp^2}$$

$$= \frac{\alpha_s(p_\perp^2)}{p_\perp^2}\frac{1}{\alpha_s(q_\perp^2)}\frac{\alpha_s(p-q)_\perp^2}{(p-q)_\perp^2}$$

Fourier transformation (up to $\ln^2$ accuracy) $\Rightarrow$

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s(|x-y|)}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}$$

$$\times\left[\frac{(x-y)^2}{X^2Y^2} + \frac{1}{X^2}\left(\frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1\right) + \frac{1}{Y^2}\left(\frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1\right)\right] \sim \alpha_s(|\Delta|_{\min})$$

Comparison with the triumvirate of Kovchegov & Weigert

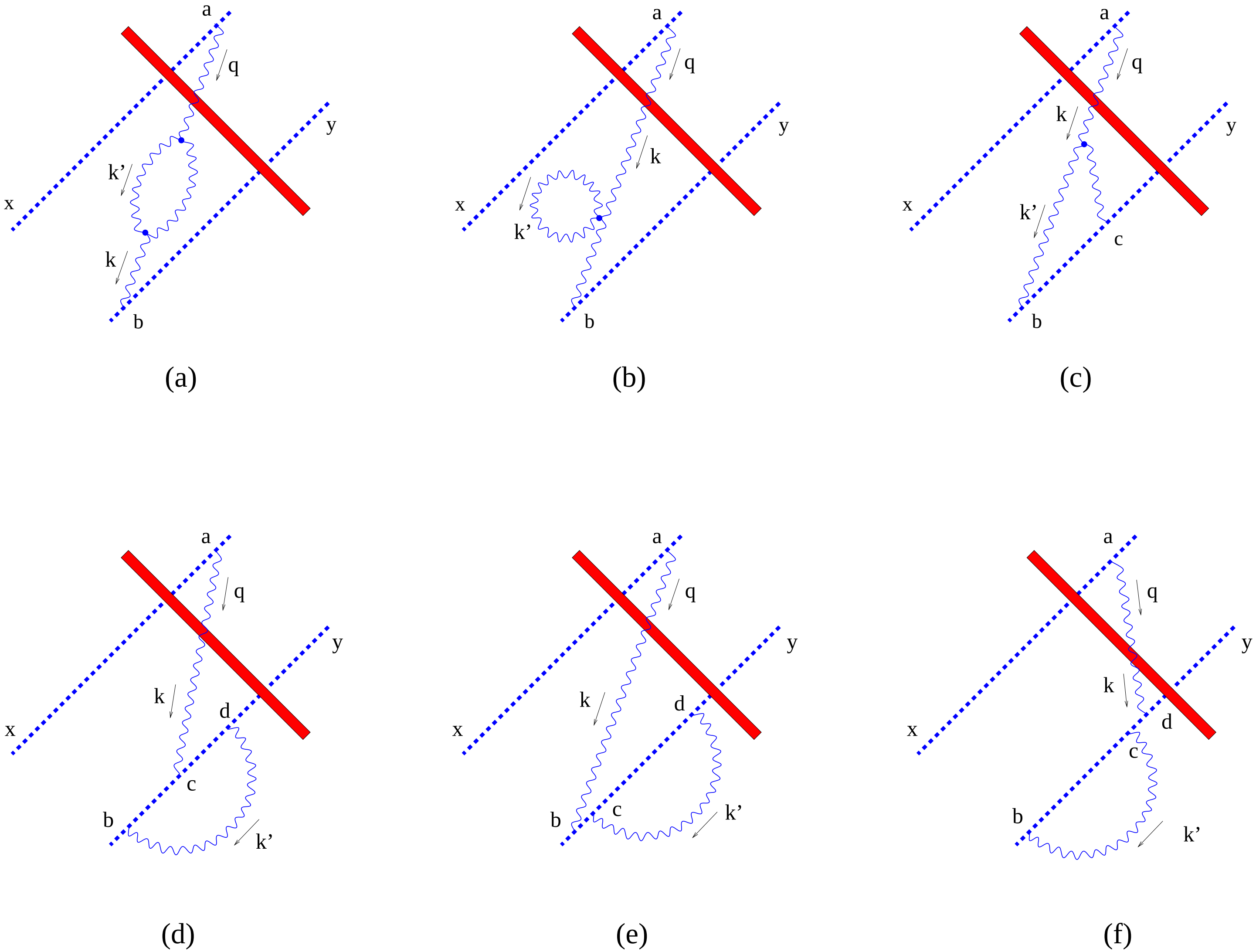
$$\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}]\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1$$
$$+\frac{b\alpha_s}{4\pi}\Big(\ln(x-y)^2\mu^2+\frac{5}{3}\Big)+\Big(\frac{b\alpha_s}{4\pi}\Big)^2\ln^2(x-y)^2\mu^2\Big] \qquad (\mathrm{B}+\mathrm{KW})$$
$$+\frac{b\alpha_s}{4\pi}\Big(\frac{1}{X^2}\ln\frac{X^2}{Y^2}\Big[1+\frac{b\alpha_s}{4\pi}\ln(x-y)^2\mu^2+\frac{b\alpha_s}{4\pi}\ln X^2\mu^2\Big]+X\leftrightarrow Y\Big)\Big\}$$

$$\Rightarrow\ \frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s((x-y)^2e^{5/3})}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}]$$
$$\times\Big[\frac{(x-y)^2}{X^2Y^2}+\frac{1}{X^2}\Big(\frac{\alpha_s(X^2)}{\alpha_s(Y^2)}-1\Big)+\frac{1}{Y^2}\Big(\frac{\alpha_s(Y^2)}{\alpha_s(X^2)}-1\Big)\Big] \qquad (\mathrm{B})$$

$$\Rightarrow\ \frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} = \frac{1}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}] \qquad (\mathrm{KW})$$
$$\times\Big[\frac{1}{X^2}\alpha_s(X^2e^{5/3})+\frac{1}{Y^2}\alpha_s(Y^2e^{5/3})-\frac{2(x-z,y-z)}{X^2Y^2}\frac{\alpha_s(X^2e^{5/3})\alpha_s(Y^2e^{5/3})}{\alpha_s(R^2)}\Big]$$

Gluon contribution to NLO BK

Sample diagrams:



Cutoff in the longitudinal momenta: $p_+ < \sigma$ for each gluon emitted by the Wilson line U_x = Pexp $ig\int A_- dx_+$

NLO kernel depends on the precise form of the cutoff.

Gluon contribution to NLO BK

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}]$$

$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(\frac{\mathbf{11}}{\mathbf{3}}\ln(\mathbf{x}-\mathbf{y})^2\mu^2+\frac{\mathbf{67}}{\mathbf{9}}-\frac{\pi^2}{\mathbf{3}}\Big)\Big]$$

$$-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{\mathbf{X^2-Y^2}}{\mathbf{X^2Y^2}}\ln\frac{\mathbf{X^2}}{\mathbf{Y^2}} - \frac{\alpha_s\mathbf{N_c}}{\mathbf{2}\pi}\frac{(\mathbf{x-y})^2}{\mathbf{X^2Y^2}}\ln\frac{\mathbf{X^2}}{(\mathbf{x-y})^2}\ln\frac{\mathbf{Y^2}}{(\mathbf{x-y})^2}\Big\}$$

$$+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}$$

$$-(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{\mathbf{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}}{\mathbf{2}(\mathbf{X'^2Y^2-Y'^2X^2})}\ln\frac{\mathbf{X'^2Y^2}}{\mathbf{Y'^2X^2}}\Big]$$

$$+[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)]$$

$$\times\Big[\frac{(\mathbf{x-y})^4}{\mathbf{X^2Y'^2(X^2Y'^2-X'^2Y^2)}}+\frac{(\mathbf{x-y})^2}{(\mathbf{z-z'})^2\mathbf{X^2Y'^2}}\Big]\ln\frac{\mathbf{X^2Y'^2}}{\mathbf{X'^2Y^2}}\Big\}\Big) =$$

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z \Big(\big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}\big]$$

$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(\frac{11}{3}\ln(\mathbf{x}-\mathbf{y})^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]$$

$$-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{\mathbf{X}^2-\mathbf{Y}^2}{\mathbf{X}^2\mathbf{Y}^2}\ln\frac{\mathbf{X}^2}{\mathbf{Y}^2} - \frac{\alpha_s N_c}{2\pi}\frac{(\mathbf{x}-\mathbf{y})^2}{\mathbf{X}^2\mathbf{Y}^2}\ln\frac{\mathbf{X}^2}{(\mathbf{x}-\mathbf{y})^2}\ln\frac{\mathbf{Y}^2}{(\mathbf{x}-\mathbf{y})^2}\Big\}$$

$$+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{\big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}$$

$$-(z'\to z)\big]\frac{1}{(z-z')^4}\Big[-2+\frac{\mathbf{X}'^2\mathbf{Y}^2+\mathbf{Y}'^2\mathbf{X}^2-4(\mathbf{x}-\mathbf{y})^2(\mathbf{z}-\mathbf{z}')^2}{2(\mathbf{X}'^2\mathbf{Y}^2-\mathbf{Y}'^2\mathbf{X}^2)}\ln\frac{\mathbf{X}'^2\mathbf{Y}^2}{\mathbf{Y}'^2\mathbf{X}^2}\Big]$$

$$+\big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)\big]$$

$$\times\Big[\frac{(\mathbf{x}-\mathbf{y})^4}{\mathbf{X}^2\mathbf{Y}'^2(\mathbf{X}^2\mathbf{Y}'^2-\mathbf{X}'^2\mathbf{Y}^2)}+\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{z}-\mathbf{z}')^2\mathbf{X}^2\mathbf{Y}'^2}\Big]\ln\frac{\mathbf{X}^2\mathbf{Y}'^2}{\mathbf{X}'^2\mathbf{Y}^2}\Big\}\Big) =$$

Running coupling part

Gluon contribution to NLO BK

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big(\big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}\big]$$
$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(\frac{11}{3}\ln(\mathbf{x}-\mathbf{y})^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]$$
$$-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{\mathbf{X}^2-\mathbf{Y}^2}{\mathbf{X}^2\mathbf{Y}^2}\ln\frac{\mathbf{X}^2}{\mathbf{Y}^2} - \frac{\alpha_s N_c}{2\pi}\frac{(\mathbf{x}-\mathbf{y})^2}{\mathbf{X}^2\mathbf{Y}^2}\ln\frac{\mathbf{X}^2}{(\mathbf{x}-\mathbf{y})^2}\ln\frac{\mathbf{Y}^2}{(\mathbf{x}-\mathbf{y})^2}\Big\}$$
$$+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{\big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}$$
$$-(z'\to z)\big]\frac{1}{(z-z')^4}\Big[-2+\frac{\mathbf{X}'^2\mathbf{Y}^2+\mathbf{Y}'^2\mathbf{X}^2-4(\mathbf{x}-\mathbf{y})^2(\mathbf{z}-\mathbf{z}')^2}{2(\mathbf{X}'^2\mathbf{Y}^2-\mathbf{Y}'^2\mathbf{X}^2)}\ln\frac{\mathbf{X}'^2\mathbf{Y}^2}{\mathbf{Y}'^2\mathbf{X}^2}\Big]$$
$$+\big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)\big]$$
$$\times\Big[\frac{(\mathbf{x}-\mathbf{y})^4}{\mathbf{X}^2\mathbf{Y}'^2(\mathbf{X}^2\mathbf{Y}'^2-\mathbf{X}'^2\mathbf{Y}^2)}+\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{z}-\mathbf{z}')^2\mathbf{X}^2\mathbf{Y}'^2}\Big]\ln\frac{\mathbf{X}^2\mathbf{Y}'^2}{\mathbf{X}'^2\mathbf{Y}^2}\Big\}\Big) =$$

Running coupling part $+$ Extra non-conformal part

Gluon contribution to NLO BK

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU^\dagger_y\} = \frac{\alpha_s}{2\pi^2}\int d^2z\, \Big(\big[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\} - N_c\mathrm{Tr}\{U_xU^\dagger_y\}\big]$$

$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(\frac{11}{3}\ln(\mathbf{x}-\mathbf{y})^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]$$

$$-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{\mathbf{X}^2-\mathbf{Y}^2}{\mathbf{X}^2\mathbf{Y}^2}\ln\frac{\mathbf{X}^2}{\mathbf{Y}^2} - \frac{\alpha_s N_c}{2\pi}\frac{(\mathbf{x}-\mathbf{y})^2}{\mathbf{X}^2\mathbf{Y}^2}\ln\frac{\mathbf{X}^2}{(\mathbf{x}-\mathbf{y})^2}\ln\frac{\mathbf{Y}^2}{(\mathbf{x}-\mathbf{y})^2}\Big\}$$

$$+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{\big[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}$$

$$-(z'\to z)\big]\frac{1}{(z-z')^4}\Big[-2+\frac{\mathbf{X}'^2\mathbf{Y}^2+\mathbf{Y}'^2\mathbf{X}^2-4(\mathbf{x}-\mathbf{y})^2(\mathbf{z}-\mathbf{z}')^2}{2(\mathbf{X}'^2\mathbf{Y}^2-\mathbf{Y}'^2\mathbf{X}^2)}\ln\frac{\mathbf{X}'^2\mathbf{Y}^2}{\mathbf{Y}'^2\mathbf{X}^2}\Big]$$

$$+\big[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_{z'}U_zU^\dagger_yU_{z'}U^\dagger_z\} - (z'\to z)\big]$$

$$\times\Big[\frac{(\mathbf{x}-\mathbf{y})^4}{\mathbf{X}^2\mathbf{Y}'^2(\mathbf{X}^2\mathbf{Y}'^2-\mathbf{X}'^2\mathbf{Y}^2)}+\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{z}-\mathbf{z}')^2\mathbf{X}^2\mathbf{Y}'^2}\Big]\ln\frac{\mathbf{X}^2\mathbf{Y}'^2}{\mathbf{X}'^2\mathbf{Y}^2}\Big\}\Big) =$$

Running coupling part $+$ Extra non-conformal part

$+$ Conformal “non-analytic” part

Gluon contribution to NLO BK

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big(\big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}\big]$$

$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]$$

$$-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}$$

$$+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{\big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}$$

$$-(z'\to z)\big]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big]$$

$$+\big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)\big]$$

$$\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big) =$$

Running coupling part + Extra non-conformal part + Conformal "non-analytic" part + Conformal analytic part

Gluon contribution to NLO BK

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}]$$

$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]$$

$$-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}$$

$$+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}$$

$$-(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big]$$

$$+[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)]$$

$$\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big) =$$

Running coupling part $+$ Extra non-conformal part

$+$ Conformal “non-analytic” part $+$ Conformal analytic part

Gluon contribution to NLO BK

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big(\big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}\big]$$

$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\big(\tfrac{11}{3}\ln(\mathbf{x}-\mathbf{y})^2\mu^2+\tfrac{67}{9}-\tfrac{\pi^2}{3}\big)\Big]$$

$$-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{\mathbf{X}^2-\mathbf{Y}^2}{\mathbf{X}^2\mathbf{Y}^2}\ln\frac{\mathbf{X}^2}{\mathbf{Y}^2} - \frac{\alpha_s N_c}{2\pi}\frac{(\mathbf{x}-\mathbf{y})^2}{\mathbf{X}^2\mathbf{Y}^2}\ln\frac{\mathbf{X}^2}{(\mathbf{x}-\mathbf{y})^2}\ln\frac{\mathbf{Y}^2}{(\mathbf{x}-\mathbf{y})^2}\Big\}$$

$$+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{\big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}$$

$$-(z'\to z)\big]\frac{1}{(z-z')^4}\Big[-2+\frac{\mathbf{X}'^2\mathbf{Y}^2+\mathbf{Y}'^2\mathbf{X}^2-4(\mathbf{x}-\mathbf{y})^2(\mathbf{z}-\mathbf{z}')^2}{2(\mathbf{X}'^2\mathbf{Y}^2-\mathbf{Y}'^2\mathbf{X}^2)}\ln\frac{\mathbf{X}'^2\mathbf{Y}^2}{\mathbf{Y}'^2\mathbf{X}^2}\Big]$$

$$+\big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)\big]$$

$$\times\Big[\frac{(\mathbf{x}-\mathbf{y})^4}{\mathbf{X}^2\mathbf{Y}'^2(\mathbf{X}^2\mathbf{Y}'^2-\mathbf{X}'^2\mathbf{Y}^2)}+\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{z}-\mathbf{z}')^2\mathbf{X}^2\mathbf{Y}'^2}\Big]\ln\frac{\mathbf{X}^2\mathbf{Y}'^2}{\mathbf{X}'^2\mathbf{Y}^2}\Big\}\Big) + \frac{\alpha_s^2N_c^2}{2\pi^2}\zeta(3)\mathrm{Tr}\{U_xU_y^\dagger\}$$

= Our result $+$ Extra term $\Rightarrow$ Coincides with NLO BFKL

Evolution of the color dipole in $\mathcal{N}=4$ SYM

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big\{[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}]$$

$$\times\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\frac{1-\pi^2}{3} - \frac{\mathbf{\alpha_s N_c}}{\mathbf{2\pi}}\ln\frac{\mathbf{X^2}}{\mathbf{(x-y)^2}}\ln\frac{\mathbf{Y^2}}{\mathbf{(x-y)^2}}\Big]$$

$$+\frac{\alpha_s}{4\pi^2}\int d^2z'[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\}$$

$$-(z'\to z)]\Big[\frac{\mathbf{(x-y)^4}}{\mathbf{X^2Y'^2-X'^2Y^2}}+\frac{\mathbf{(x-y)^2}}{\mathbf{(z-z')^2}}\Big]\frac{\ln\mathbf{X^2Y'^2/X'^2Y^2}}{\mathbf{X^2Y'^2}}\Big\} =$$

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$$-\,(z'\to z)]\Big[\frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}+\frac{(x-y)^2}{(z-z')^2}\Big]\frac{\ln X^2Y'^2/X'^2Y^2}{X^2Y'^2}\Big\} =$$

Non-conformal part

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Non-conformal part $+$ Conformal part

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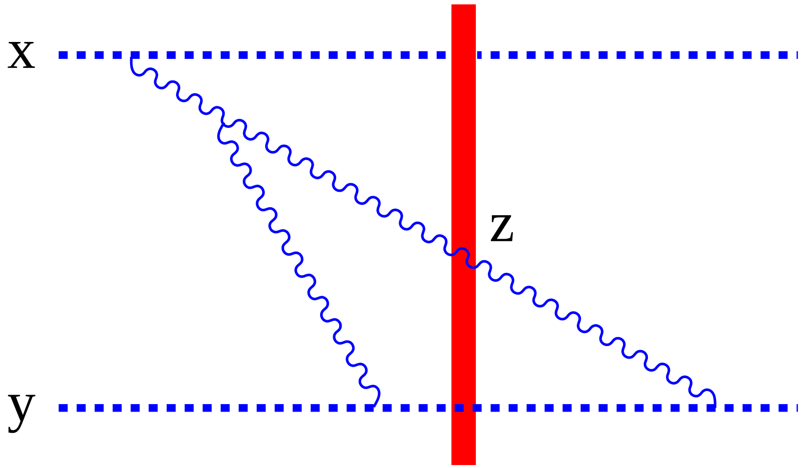
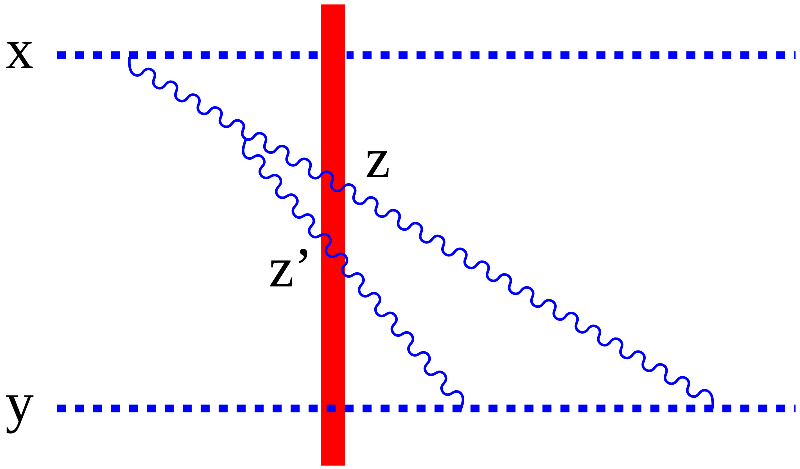
Non-conformal part
("recombination of dipoles")



$+$

Conformal part
("creation of dipoles")



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$$\times\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\frac{1-\pi^2}{3} - \frac{\alpha_s N_c}{2\pi}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big]$$

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$$-(z'\to z)]\Big[\frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}+\frac{(x-y)^2}{(z-z')^2}\Big]\frac{\ln X^2Y'^2/X'^2Y^2}{X^2Y'^2}\Big\} + \frac{\alpha_s^2N_c^2}{2\pi^2}\zeta(3)\mathrm{Tr}\{U_xU_y^\dagger$$

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Conclusions

- High-energy scattering can be described in terms of dipoles (Wilson lines) $U_x U_y^\dagger$ - no new operators at the 1-loop level (for gluon contributions, such possible terms = difference between $\frac{67-3\pi^2}{9}$ and the NLO BFKL terms = 0).
- For the creation of dipoles in the small-x evolution, the coupling constant is determined by the size of the smallest dipole.
- The NLO evolution kernel depends on the precise definition of the cutoff in the longitudinal momenta.
- With $|\alpha| < \sigma$ cutoff, the NLO BK/NLO BFKL for $\mathcal{N} = 4$ SYM is *almost* conformally invariant in the transverse plane.

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Outlook

- An extra $\zeta(3)$ as compared to NLO BFKL - different cutoff in α?
- $\mathcal{N} = 4$: Is there operator/regularization such that the kernel is conformal?

The calculation of gluon part is done in collaboration with my student G. Chirilli.