Chapter 1

PROBING THE QCD PHASE BOUNDARY WITH FINITE COLLISION SYSTEMS

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1. INTRODUCTION

This article treats three closely-related topics. The common theme is the degree to which classical thermodynamic and statistical concepts are applicable to small-number dynamical systems near a bulk-matter phase boundary.

I begin with an examination of finite-size effects and the observability of a proposed QCD tricritical point. In contrast to macroscopic bulk-matter behavior a finite system cannot exhibit dramatic variation of order-parameter dependence on system control parameters (*e.g.*, first-order phase transition). Predictions of novel structure on a phase boundary should include an estimate of observability based on available system size and statistical realities.

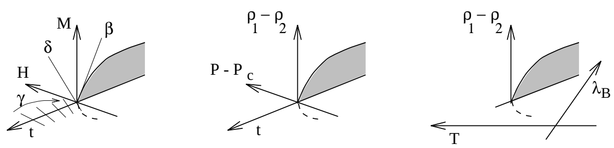
I then consider the relationship between linear response coefficients (*e.g.*, heat capacity) and fluctuations in local extensive quantities. Non-statistical fluctuations are proposed as a signature of the QGP phase transition. Gaussian fluctuation theory implies that fluctuation amplitudes can be represented by linear response coefficients. This description fails near a phase boundary.

Finally, I consider application of the Central Limit Theorem to event-by-event distributions of global variables. Detailed examination of a proposed differential measure based on the CLT reveals that there are a number of contributions to such a measure, making interpretation difficult. Direct study of more elementary differential measures provides a better path to the underlying physics.

2. OBSERVABILITY OF THE QCD TRICRITICAL POINT

General arguments based on universality have recently been put forward in support of a localized structure on the QCD phase boundary: a tricritical point or critical endpoint located at some intermediate position in the (T, μ_B) plane [1]. Detection and study of such a structure would be an important contribution to our understanding of full QCD. From an experimental viewpoint the question immediately arises what affect finite system size or scale interval has on the observability of such a structure?

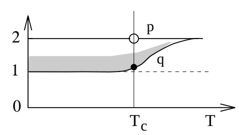
The relevant scale interval is bounded by the characteristic size a of a typical momentum-carrying object and the characteristic size L of a 'causally connected' or 'equilibrated' region. The ratio L/a then determines the maximum sharpness of structures near critical points. The effective scale interval is a function of space and time in the collision space-time volume. We can use 'system size' informally to mean some average or effective L/a where this is not confusing.



$\Phi \sim t^p H^q \quad p = 2 \;\; q = 6/5$

	α	β	γ	δ
(p,q)	0	1/3	4/3	5
exp.	0.1	1/3	4/3	4-5

d	2	3	4	exp.
η	1/4	0	0	0.1
ν	1	2/3	1/2	2/3



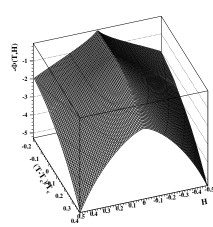


Figure 1.1 Three instances of a critical point - Ising model, liquid-gas phase boundary and proposed QCD critical endpoint (top). Critical exponents corresponding to parameter pair $(p, q) = (2, 6/5)$ (middle). Schematic temperature dependence of exponent parameters and corresponding free-energy surface (bottom).

The tricritical-point proposal suggests varying 'control parameters' (state variables) as a method of study. Practically accessible control parameters are collision geometry, energy and nuclear collision partners. These parameters map in some loose way onto thermodynamic equivalents (T, μ_B). However, the relationship between these two parameter systems is itself a subject of intense study and may always be poorly defined due to lack of complete system equilibration.

To explore the consequences of finite system size on critical phenomena the Ising magnet serves as a paradigm. Near the critical point the standard set of six critical exponents are linearly interdependent through

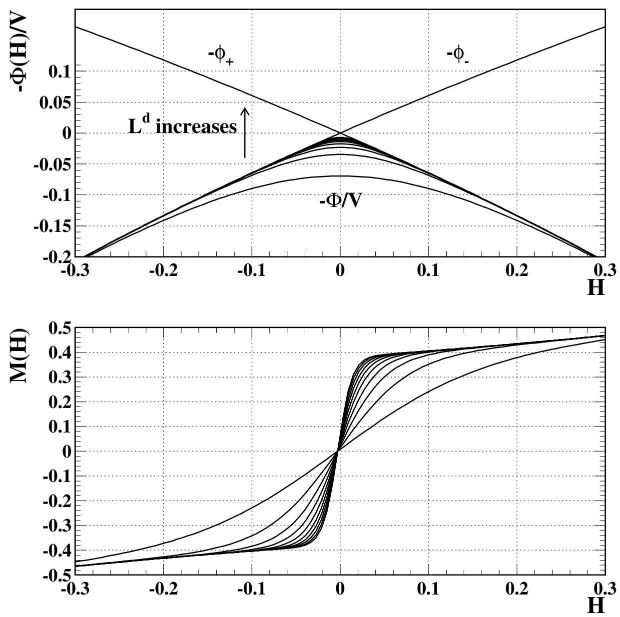


Figure 1.2 Sketch of free-energy dependence on external field for Ising model and various system sizes L. In the asymptotic limit $L \to \infty$ one recovers a first-order transition.

a system of scaling laws and can be reduced to two independent quantities (say (p,q)). These quantities represent the compound curvature of the free-energy surface in the neighborhood of the critical point along the control parameters T and H respectively. For this universality class the critical exponents are well represented by $(p,q) = (2, 6/5)$ as shown in Fig. 1.1.

How does finite system size affect this picture? Considering the discontinuity in $M(H)$ for bulk matter below the critical temperature, we can express the total free-energy density in terms of the individual free-energy densities ϕ_+ and ϕ_- of the two spin populations. The overall free-energy density can be written

$$\Phi(H) \quad \approx \quad -log\left\{e^{-\phi_+(H)\cdot(L/a)^d} + e^{-\phi_-(H)\cdot(L/a)^d}\right\}\cdot(a/L)^d \qquad (1.1)$$

If $L/a \to \infty$ and given the nearly linear dependence of $\phi_\pm$ on H near the origin we have

$$\Phi(H) \quad \approx \quad |H|^1 \qquad (1.2)$$

for $|H| \approx 0$, at and below the critical temperature, leading to an asymptotic (in scale interval) singularity in the magnetic susceptibility *vs* external field. The functional form of $M(H)$ therefore depends parametrically on the scale interval as $(L/a)^d$. This dependence is illustrated in Fig. 1.2. Thus, the consequence of finite system size is the general elimination of sharp structures on a phase boundary. If the QCD phase boundary is 'observed' with a finite collision system the manifestation of a tricritical point may be undetectable because of finite-scale-interval effects.

3. FLUCTUATION THEORY NEAR A CRITICAL POINT

Critical phenomena include nonstatistical fluctuations in local extensive quantities. Fluctuation characteristics are determined by the dependence of the free energy on relevant state variables. The standard treatment of fluctuations is the Einstein/gaussian theory or linear response theory [2]. This treatment cannot predict fluctuation properties in the neighborhood of a critical point.

The state of a large but bounded dynamical system in equilibrium can be characterized by a set of global intensives $\vec{\alpha} = (T, \lambda, P, ...)$, the system control parameters. If this system is randomly partitioned the partition elements can be characterized by local extensives $X_\alpha = (E, N, V, ...)$. An equilibrated system can be further characterized by a free energy $\Phi = (TS, A, G, ...)$. A local extensive can then be expressed in terms of its conjugate intensive and a free energy as $X_\alpha = \frac{\partial \Phi}{\partial \alpha}$. Linear response coefficients χ_α (C_V, κ_T,...) can be defined by

$$\begin{aligned} \chi_\alpha &= \frac{\partial X_\alpha}{\partial \alpha} \\ &= \frac{\partial^2 \Phi}{\partial \alpha^2} \end{aligned} \tag{1.3}$$

The basic consequence of the gaussian fluctuation model (free energy assumed gaussian in the neighborhood of its maximum) is that a fluctuation variance is directly proportional to a corresponding linear response coefficient

$$\begin{aligned} (\cdot)\sigma^2_{X_\alpha} &= (\cdot)\frac{\partial X_\alpha}{\partial \alpha} \\ &= N \cdot \frac{\partial log\{X_\alpha\}}{\partial log \alpha} \end{aligned} \tag{1.4}$$

where $(\cdot)$ is defined by requiring that $(\cdot)X^2_\alpha \approx N^2$.

If the density of DoF (degrees of freedom) near a phase boundary is represented as the product of a fugacity and a singular part $\rho(\alpha) = \lambda \cdot g(\alpha)$ then a typical variance density can be written as

$$(\cdot)\sigma^2_{X_\alpha}/V = \rho(\alpha)\{1 + \hat{\alpha} \cdot \nabla_\alpha log[g(\alpha)]\} \tag{1.5}$$

where ∇_α is a logarithmic gradient with respect to the global intensives $\vec{\alpha}$ and $\hat{\alpha}$ is a unit vector in the intended direction of parameter variation. Far from a phase boundary, where gaussian theory is applicable

$$\{1 + \hat{\alpha} \cdot \nabla_\alpha log[g(\alpha)]\} \;\approx\; C_V/N,\; P \cdot \kappa_T,\; V \cdot \alpha_t, ... \tag{1.6}$$

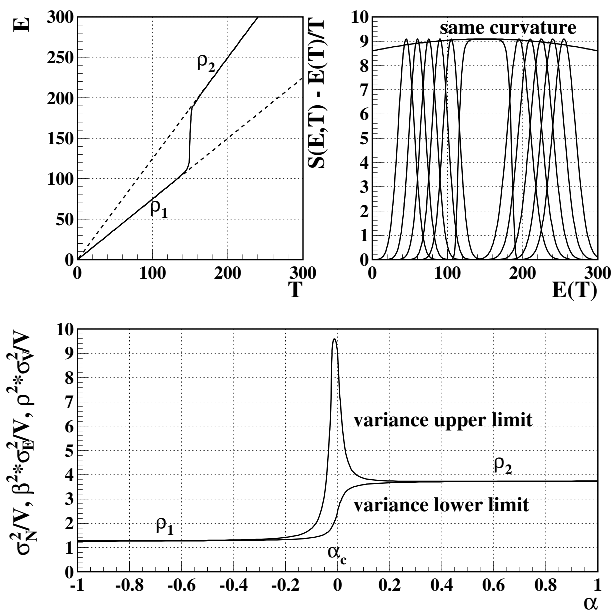


Figure 1.3 Sketch of E(T) dependence near a phase transition (top left), fluctuation distribution for various temperatures (top right) and plot of fluctuation amplitude vs temperature estimated by linear response coefficients (upper limit) and DoF density (lower limit) (bottom).

These relationships are represented in Fig. 1.3. The $E(T)$ dependence is piece-wise linear, corresponding to two different density regions separated by a phase boundary. At top right is a representation of energy fluctuation distributions at various temperatures. Near the critical temperature the fluctuations increase in amplitude but are bounded, whereas the linear response coefficient (slope of $E(T)$) may be arbitrarily large. The apparent paradox then is that linear response coefficients may become arbitrarily large in the neighborhood of a critical point but the observed fluctuation amplitude may not be large. Gaussian fluctuation theory would predict

$$\sigma^2_{X_\alpha} \propto \chi_\alpha \quad \rightarrow \quad \infty \tag{1.7}$$

when fluctuations are in fact bounded.

The paradox is resolved by noting that linear response coefficients measure the *curvature* of the free-energy surface near its maximum. If the free-energy surface is approximately gaussian in the neighborhood of its maximum then Eq. (1.4) is a good approximation. However, near a critical point the free-energy curvature and the width may have *no well-defined relationship*. The gaussian treatment breaks down and linear response coefficients cannot predict or represent fluctuation amplitudes. Near a phase boundary a separate determination must be made of the local variance density and the linear response coefficients, to the extent that the latter are meaningful or useful.

In the bottom panel of Fig. 1.3 the upper curve represents an estimate of fluctuation variance density based on linear response coefficients and serves as an upper limit in the neighborhood of a phase boundary. The lower curve is the density of DoF and serves as a lower limit to the variance density. Actual behavior near the phase boundary depends on the detailed structure of the free-energy surface. Both linear response coefficients and fluctuation amplitudes are in principle measurable. Instead of being redundant (as in gaussian theory) they may actually provide complementary information about the underlying system Hamiltonian.

4. ANALYSIS OF MEAN-P$_T$ FLUCTUATIONS

Various event-wise global variables can be extracted from a collision final-state multiparticle momentum distribution. The most straightforward are the moments of the p_t distribution for charged hadrons, especially the first moment $<p_t>$. Preliminary analysis of the $<p_t>$ distribution for 100k central Pb-Pb events has been completed by the NA49 collaboration [3]. To summarize, it is found that this distribution is very close to a simple gaussian. The question arises whether this distribution is consistent with simple finite-number statistics modified by known hadronic physics, or whether the distribution contains evidence at some level for nonstatistical fluctuations that one might expect near a QCD phase boundary. How does one formulate this problem in a quantitative way?

The basis for forming a differential measure applicable to this problem is the Central Limit Theorem (CLT). The CLT can be formulated in the

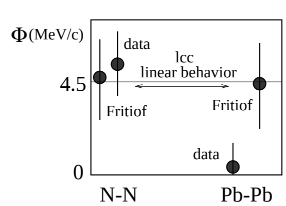


Figure 1.4 Summary of NA49 Φ analysis results.

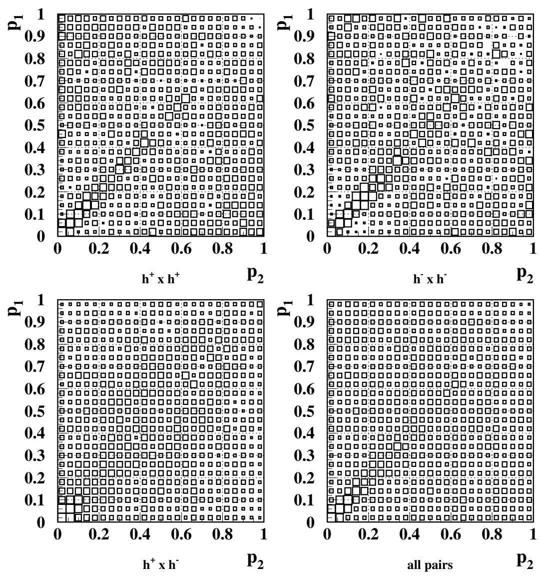


Figure 1.5 Distribution of sibling/mixed pair-density ratio on $p_t \otimes p_t$ space for 158 GeV/c Pb-Pb collisions. Density-ratio variation (box size) is within the interval $1 \pm .005$.

following way. If events of N *independent* p_t samples with total momentum P_t are drawn from a *static* parent p_t distribution the distribution of event means $<p_t> = P_t/N$ approximates a gaussian distribution with mean approaching the parent mean and variance $\sigma^2_{P_t}$ related to the parent variance $\sigma^2_{p_t}$ by

$$\sigma^2_{P_t} \quad = \quad N \cdot \sigma^2_{p_t} \tag{1.8}$$

A related differential width measure, the so-called Φ_{p_t} measure , has been proposed [4] and used to analyze the NA49 $<p_t>$ distribution for 100k events [5]. It has the basic structure

$$\Phi_{p_t} \quad = \quad \sqrt{\sigma^2_{P_t}/N} - \sqrt{\sigma^2_{p_t}} \tag{1.9}$$

and is zero if the event population is consistent with the CLT.

Departures from the CLT ($\Phi_{p_t} \neq 0$) are due to 1) (anti)correlated samples and 2) variation of the parent during the sample history. Net positive correlation implies that $N_{eff} < N$, that is, the *effective* sample is less than N. We then expect $\Phi_{p_t} > 0$. The opposite is true for net anticorrelated samples. If the parent distribution varies during the sampling it can be shown that $\Phi_{p_t} \approx \sigma^2_{\overline{p_t}}$, where $\sigma^2_{\overline{p_t}}$ is a variance measure for the (slowly-varying) parent-distribution centroid.

Preliminary results of the NA49 Φ_{p_t} analysis which compares N-N and Pb-Pb collision events are summarized in Fig. 1.4 and seem to indicate that correlations observed in N-N collisions are reduced or eliminated, possibly by rescattering, in Pb-Pb collisions [6]. In order to provide a clearer understanding of the Φ_{p_t} measure and offer improved insight into the underlying physics we can decompose Φ_{p_t} in the following way

$$\Phi_{p_t} \approx \frac{A^2 + B^2}{2\sigma_{p_t} \cdot \overline{N}} \tag{1.10}$$

where

$$\begin{aligned} A^2 &= \sum_{i \neq j} p_i p_j - \overline{N_e(N_e-1)} \cdot \overline{p_t}^2 \\ B^2 &= -2\overline{p_t}\left(\overline{N_e^2 < p_t >_e} - \overline{N_e^2} \cdot \overline{p_t}\right) \end{aligned} \tag{1.11}$$

A^2 is effectively a weighted integral of net two-particle correlation, and B^2 is a measure of N-<p$_t$> correlations.

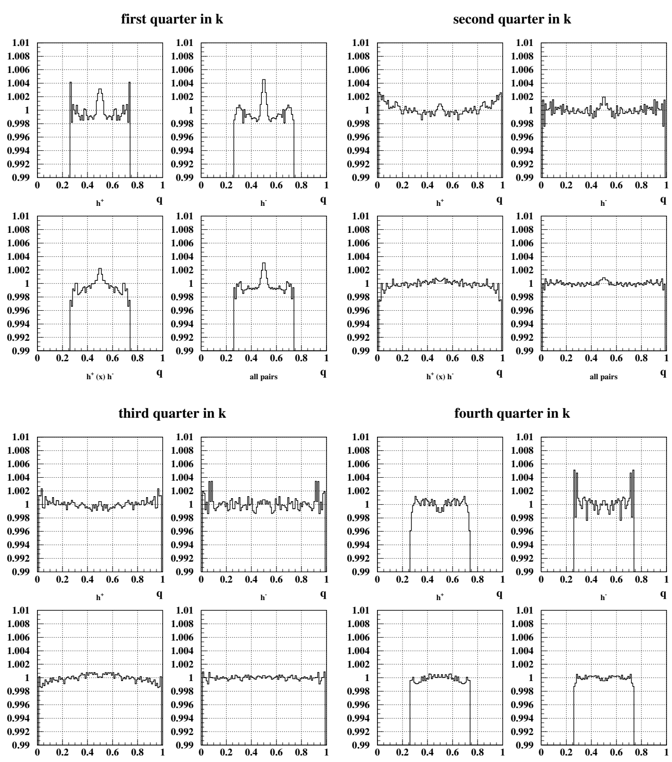


Figure 1.6 Projection of pair-density ratio on pair momentum difference q for four bins in mean pair momentum k.

Contributions to A^2 include but are not limited to quantum statistics, Coulomb effects, particle decays, instrumental effects and true dynamical fluctuations. We study this term in more detail by forming a two-particle space (p_1, p_2) for sibling pairs and mixed pairs. We plot the ratio of density distributions ρ_{sib} and ρ_{mixed} for sibling and mixed pairs, and also project these densities onto momentum difference q and mean momentum k by analogy with standard HBT analysis. The connection of ρ_{sib} and ρ_{mixed} to A^2 is given by

$$A^2 \approx \int p_1 p_2 \cdot \{\rho_{sib} - \rho_{mixed}\} dp_1 dp_2 \tag{1.12}$$

The density ratio ρ_{sib}/ρ_{mixed} is shown in Fig. 1.5 for two combinations of like-sign pairs, one of unlike-sign pairs and one for all pairs taken together. One observes a prominent ridge along the main diagonal corresponding to quantum statistics and a peak at low k in the unlike-signed pairs due to the Coulomb interaction. We can project these densities onto q for four k slices and take the ratio of projections as shown in Fig. 1.6.

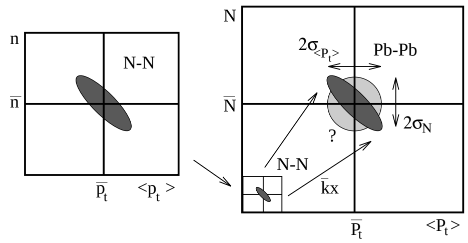
One sees quite prominently the expected correlation peak at small q and k due to BE statistics, but there is also a very strong *anti*correlation at large q in the like-signed pair distributions and a strong correlation at large q in the unlike-signed pair distribution. These results are unexpected and increase the difficulty of interpreting the Φ_{p_t} analysis. Certainly a separate treatment of positive, negative and unlike-signed pairs is necessary in order to extract complete information. Different physics may affect each combination.

Contributions to B^2 come from correlations between event multiplicity and $<\mathrm{p}_t>$. In N-N collisions these correlations are well-known and understood as a manifestation of energy conservation. The correlation is significantly different for negative and positive hadrons. The results of direct calculation of the linear correlation coefficient (lcc) between N and $<\mathrm{p}_t>$ are summarized in Fig. 1.7. One sees that even for Pb there is a very significant nonzero lcc for total hadrons. The correlation is not completely removed by final-state rescattering but remains measurable.

For the NA49 Φ_{p_t} analysis it is found that A^2 and B^2 make contributions to Φ_{p_t} which are comparable in magnitude ($\simeq 20 MeV/c$) and opposite in sign. This may be a case where different physical phenomena conspire to produce a small value for a given composite measure.

5. CONCLUSIONS

Interpretation of event-by-event studies of relativistic heavy-ion collisions requires an approach to statistical and thermodynamical measures



lcc	h	h^+	h^-	Φ(MeV/c)
N-N	[-0.09] h+/- average	-0.06 ± 0.02	-0.12 ± 0.02	4.5 ± 2.2
Pb-Pb	-0.031 0.003			0.1± 0.7

Figure 1.7 Sketch of N-$<p_t>$ linear correlation for N-N (top left), Pb-Pb (top right) and summary of measured values of the corresponding linear correlation coefficient (bottom).

which includes limitations of finite system size and departures from classical statistics. The issue of critical-point phenomena on finite systems is interesting in its own right, and must be an important component of the study of the QCD phase boundary. The relationship between fluctuation variances and linear response coefficients in the neighborhood of the phase boundary is also important. Deviations from gaussian theory may in fact provide additional information about the free-energy surface and full QCD. Statistical analysis of global momentum-space variables requires differential analysis techniques which are capabable of separating several competing physical phenomena contributing to the correlation structure of the final state. The Φ_{p_t} measure represents an important step in the study of event-by-event fluctuations. Elaborations of this measure are now required for more detailed studies.

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