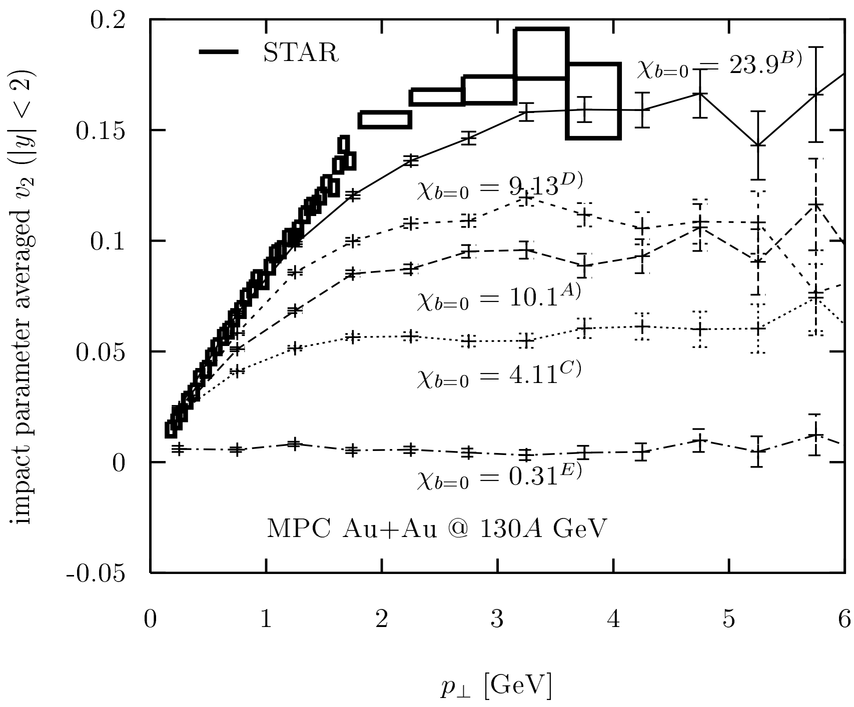
Solution of the Boltzmann Equation: Denes Molnar

1000 Classical Massless Particles with Constant X-Sections

$\sigma_0 \approx 10 - 20\,$mb



- The Boltzmann equation predicted a flattening of v_2 at high p_T.
- The observed v_2 breaks down in a way consistently with viscous/Boltzmann effects.
- Why do the cross sections need to be

HUGE!??

Viscous corrections to Ideal Hydrodynamics:

$$T^{\mu\nu} = T_o^{\mu\nu} + T_{vis}^{\mu\nu}$$

where

$$T_{vis}^{ij} \equiv \eta \left\langle \nabla^i u^j \right\rangle \sim \eta \left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\, \partial_l v^l\right)$$

For a Bjorken expansion we have:

$$T_{vis}^{zz} \sim \eta \partial^z v^z \sim -\frac{\eta}{\tau}$$

Doing the algebra:

$$\begin{aligned} T^{\mu\nu} &= T_o^{\mu\nu} + T_{vis}^{\mu\nu} \\ &= \begin{pmatrix} \epsilon & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} + \begin{pmatrix} 0 & & & \\ & \frac{2}{3}\frac{\eta}{\tau} & & \\ & & \frac{2}{3}\frac{\eta}{\tau} & \\ & & & -\frac{4}{3}\frac{\eta}{\tau} \end{pmatrix} \end{aligned}$$

- The Longitudinal Pressure is reduced by $\frac{4}{3}\eta/\tau$.
- The Transverse Pressure is increased by $\frac{2}{3}\eta/\tau$.

Expect p_T spectra to be pushed out to larger p_T

- In a Radially Symmetric way

How valid is Hydro? How much Entropy is produced?

$$\frac{d(\tau s)}{d\tau} = 0 \qquad \text{(Ideal Case)}$$
$$\frac{d(\tau s)}{d\tau} = \frac{\frac{4}{3}\eta+\sigma}{\tau T} \qquad \text{(Viscous Case)}$$

For hydrodynamics to be valid, the entropy produced over the time scale of the system, τ , must be small compared to the total :

$$\tau\,\frac{\frac{4}{3}\eta+\sigma}{\tau T}\,\frac{1}{\tau s} \equiv \Gamma_s\frac{1}{\tau} \ll 1$$

where $\Gamma_s \equiv \frac{\frac{4}{3}\eta+\sigma}{e+p}$ is the *Sound Attenuation Length*. Note ($sT = e + p$).

Estimates of Shear Viscosity:

$$\underbrace{T^{00} + T^{zz}}_{\text{indep. of Bag Const.}} = (e+p) + \eta\,\langle\partial^z v^z\rangle$$

$$\sim (e+p)\left[1 + \underbrace{\ell_{m.f.p.}}_{1/(n\sigma_0)}\langle\partial^z v^z\rangle\right]$$

Using $e + p \sim nT$, then find:

$$\Gamma_s \sim \ell_{m.f.p.} \quad \text{and} \quad \eta \sim \frac{T}{\sigma_0} \quad \text{and} \quad \frac{\Gamma_s}{\tau} \sim \frac{\ell_{m.f.p.}}{\tau} \ll 1$$

Estimates of η for the QGP and Heavy Ion Collisions

Perturbative QCD – Arnold, Moore, Yaffe.

- $\eta \approx 150\; T^3 \, \frac{1}{g^4}$.

  Based upon kinetic theory of quarks and gluons.

  Set $\alpha_s \to 1/2$ and $m_D \to$ a reasonable value

$$\left(\frac{\Gamma_s}{\tau}\right) \approx 0.4 \frac{1}{\tau T}$$

Strongly Coupled conformal N=4 SYM – Son, Starinets, Policastro

- No kinetic theory exists. Like most real liquids.

$$\left(\frac{\Gamma_s}{\tau}\right) = \frac{1}{3\pi} \frac{1}{\tau T} \approx 0.11 \frac{1}{\tau T}$$

Phenomenology – Molnar

Found could fit elliptic flow $v_2(p_T)$ only when

- $\frac{dN}{d\eta} = 1000$, $\sigma_0 = 10 \div 20$, and $\tau_o = 0.1$ fm.

$$\Gamma_s = 0.421 \frac{1}{n\sigma_0} \qquad \left(\frac{\Gamma_s}{\tau}\right) = 0.02 \div 0.04$$

- Constant cross section. Independent of time!

Best Guess: (At time τ_0)

With

$$T_o \sim 300\,\text{MeV} \quad \text{and} \quad \tau_0 \sim 1\,\text{fm}$$

Find:

$$\left(\frac{\Gamma_s}{\tau}\right) \approx 0.1 - 0.4$$

How does $\frac{\Gamma_s}{\tau}$ evolve?

Thermalization: How does Γ_s/τ evolve?

1. Bjorken Expansion | Scale Invariant Cross Section: $\sigma \sim \frac{\alpha_s^2}{T^2}$

- When Entropy is conserved: $T \sim \frac{1}{\tau^{1/3}}$

$$\frac{\Gamma_s}{\tau} \sim \frac{\#}{\tau T} \sim \# \frac{1}{\tau^{2/3}}$$

- When $\frac{dE}{dy}$ is conserved (entropy is produced): $T \sim \frac{1}{\tau^{1/4}}$

$$\frac{\Gamma_s}{\tau} \sim \frac{\#}{\tau T} \sim \# \frac{1}{\tau^{3/4}}$$

$\Longrightarrow$ rapid to rapid++ thermalization

2. Bjorken Expansion | Constant Cross Section: $\sigma = \sigma_0$

- When Entropy is Conserved: $\tau n \sim$ Const

$$\frac{\Gamma_s}{\tau} \sim \frac{\ell_{m.f.p.}}{\tau} \sim \frac{1}{\tau n \sigma_0} \sim \text{Const}$$

- When $\frac{dE}{dy}$ is conserved (entropy is produced): $\tau n \sim \tau^{1/4}$

$$\frac{\Gamma_s}{\tau} \sim \frac{\ell_{m.f.p.}}{\tau} \sim \frac{1}{\tau n \sigma_0} \sim \frac{1}{\tau^{1/4}}$$

$\Longrightarrow$ Constant to slow thermalization

Spherical Expansion | Scale Invariant Cross Section: $\sigma \sim \frac{\alpha_s^2}{T^2}$

- Entropy conservation: $(sV) \sim$ Const and $s \sim T^3$. Then $T \sim \frac{1}{\tau}$.

$$\frac{\Gamma_s}{\tau} \sim \frac{\#}{\tau T} \sim \text{Const}$$

- Energy conservation: $(eV) \sim$ Const and $e \sim T^4$ $T \sim \frac{1}{\tau^{3/4}}$

$\Longrightarrow$ Constant to slow thermalization

Spherical Expansion | Constant Cross Section: σ_0

- Entropy Conservation: $(nV) \sim$ Const. $n \sim \frac{1}{\tau^3}$

$$\frac{\Gamma_s}{\tau} \sim \frac{\ell_{m.f.p.}}{\tau} \sim \frac{1}{\tau\, n \sigma_0} \sim \frac{\tau^2}{\sigma_0}$$

- Energy Conservation: $(eV) \sim$ Const. and $n \sim T^3 \sim 1\tau^{9/4}$

$$\frac{\Gamma_s}{\tau} \sim \frac{\ell_{m.f.p.}}{\tau} \sim \frac{1}{\tau\, n \sigma_0} \sim \frac{\tau^{5/4}}{\sigma_0}$$

$\Longrightarrow$ rapid++ to rapid breakup.

Summary:

σ	1 D Expansion	3 D Expansion
$\frac{\alpha_s}{T^2}$	++ $\sim \frac{1}{\tau^{2/3}}$	Const.
σ_0	Const.	-- $\sim \frac{\tau^2}{\sigma_0}$

Three models of viscosity:

- A scale free model: $\eta \propto T^3$

$$\eta = \frac{1}{5}\, s$$

- Constant cross section:

$$\eta = 1.2\, \frac{T}{\sigma_0} \text{ with } \sigma_0 = 10\,\text{mb.}$$

- A minimal model: $e_c = 1\,\text{GeV/fm}^3$

$$\eta = \begin{cases} 1.2\,\frac{\mathrm{T}}{\sigma_0} & \text{for } e < e_c \\ \frac{1}{5}\,\mathrm{s} & \text{for } e > e_c \end{cases}$$

Solving the Relativistic Navier Stokes Equations RNSE

- The RNSE as written can not be solved.

  There are unstable modes which propagate faster than the speed of light.

- Why? Because the stress RNSE tensor is not allowed time to change.

$$T^{ij}_{vis}\Big|_{\text{instantly}} = \eta\left(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\partial_i v^i\right)$$

- Can make many models which relax to the RNSE.

  (Drude, Maxwell, P.C. Martin, L. Lindblom, R. Geroch)

$$T^{ij}_{vis}\Big|_{\omega\to 0} \sim \eta\left(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\partial_i v^i\right)$$

- In the regime of validity of hydrodynamics the models all agree with each other and with RNSE.

Can solve these models

A simple model:

(Inspired by H.C. Ottinger, Physica 1997)

Imagine a tensor c_{ij} which relaxes quickly to $\partial_i v_j + \partial_j v_i$

$$\partial_t c_{ij} - (\partial_i v_j + \partial_j v_i) = \frac{\bar{c}_{ij}}{\tau_0} + \frac{\langle c_{ij} \rangle}{\tau_2}$$

where $\bar{c}_{ij} = (tr\,\mathbf{c})\,\delta_{ij}$ and $\langle c_{ij} \rangle = c_{ij} - \frac{1}{3}\bar{c}_{ij}$

For small τ_0 and τ_2 we have:

$$c_{ij} \approx \tau_0 \delta_{ij}\,\partial_i v^i + \tau_2(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_l v^l)$$

Then the "effective" pressure for small strains is given by:

$$T_{ij} \approx p(\delta_{ij} - a_1\, c_{ij})$$

Comparing this to the canonical form:

$$T_{ij} \approx p\delta_{ij} + \sigma\partial_i v^i + \eta(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_l v^l)$$

Can map: $(\tau_0, \tau_2\,, a_1) \to (\sigma, \eta, c_\infty)$

Another Model:

(Inspired by Lindblom and Geroch, Phys. Dev. D1994)

Write a set conservation/balance laws:

$$\begin{aligned}
\partial_\mu(N^\mu) &= 0 \\
\partial_\mu(T^{\mu\nu}) &= 0 \\
\partial_\mu(A^{\mu\alpha\beta}) &= I^{\alpha\beta}
\end{aligned}$$

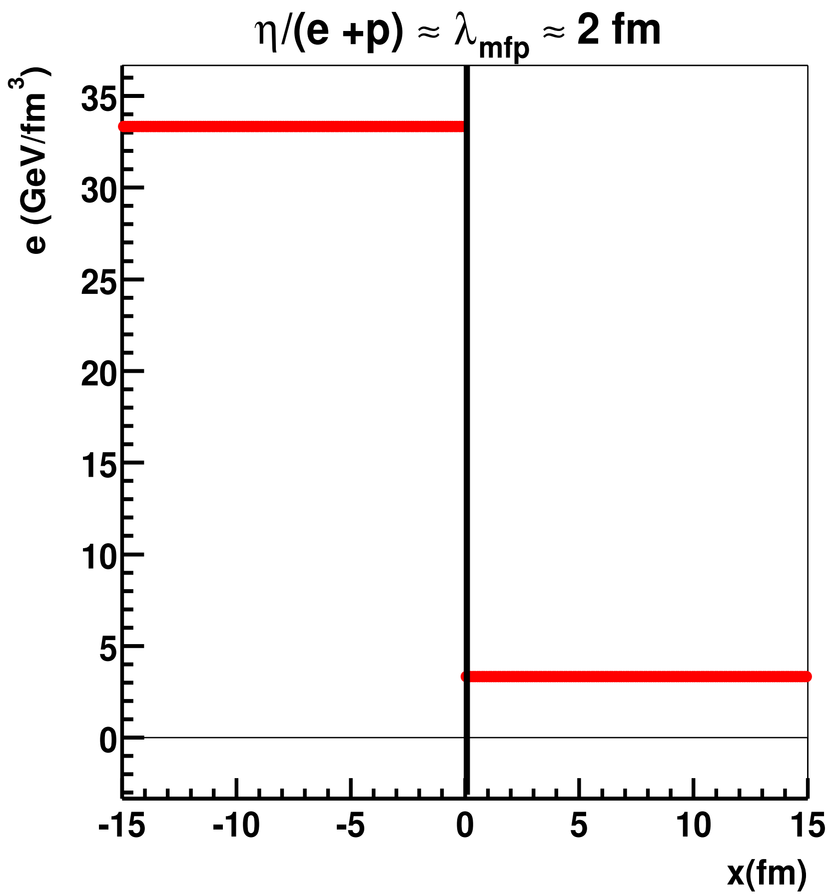
where

$$\begin{aligned}
N^\mu &= n u^\mu \\
T^{\mu\nu} &= e u^\mu u^\nu + p\Delta^{\mu\nu} + u^\mu q^\nu + u^\nu q^\mu + \tau^{\mu\nu} \\
A^{\mu\alpha\beta} &= 2T\Delta^{\mu(\alpha}u^{\beta)} \\
I^{\alpha\beta} &= -\frac{T}{\eta}\tau^{\alpha\beta} - \frac{2T}{3\sigma}\Delta^{\alpha\beta} - \frac{2T}{\kappa T}\left(q^\alpha u^\beta + q^\beta u^\alpha\right)
\end{aligned}$$

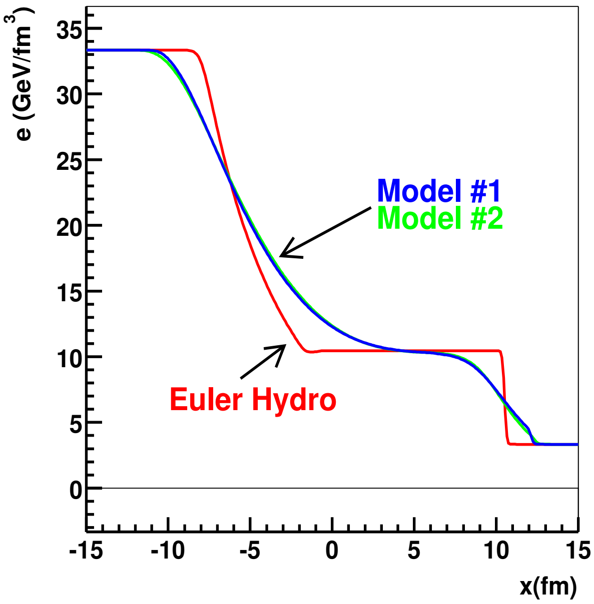
- A completely different model at short times
- Only the long time behavior is the same.

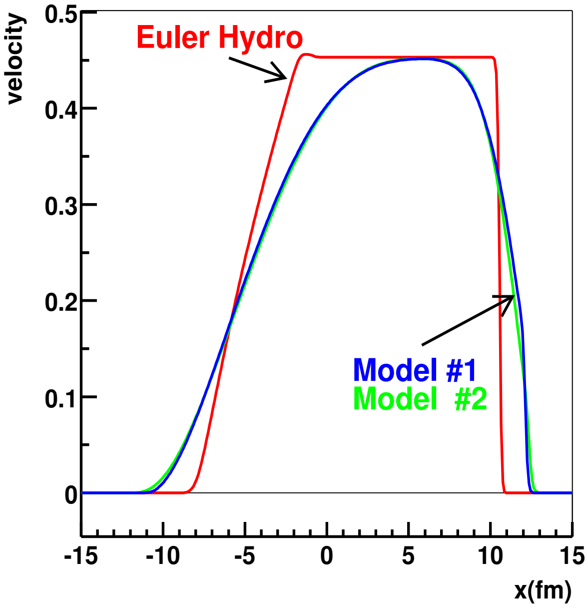
  The long time behavior is controlled by the viscous coefficients.
- None of the details of these models should matter.

Sod's Test Problem



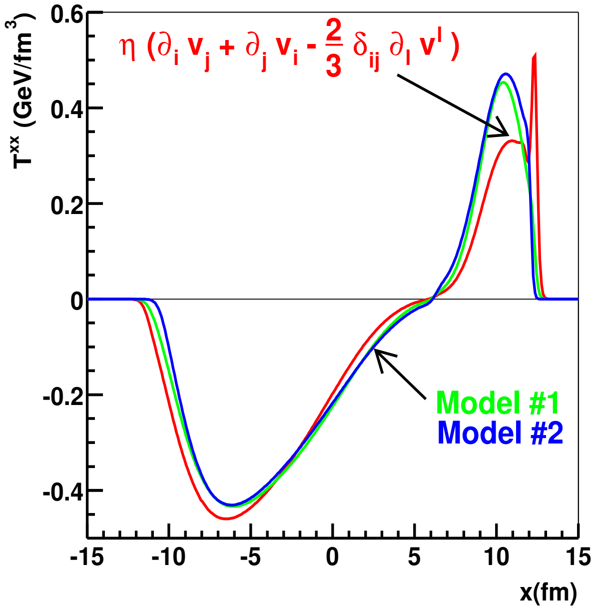
Compare the different models:





Compare the stress tensor with the form expected of the

Navier Stokes Equations:



Summary

- All models agree about the solution to the Navier Stokes equations.
- The stress energy tensor is almost always very close to

$$T^{ij} \sim \eta \left(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\partial_l v^l \right)$$

This holds in the domain of validity of hydrodynamics.

When the viscous term is about half of the pressure :

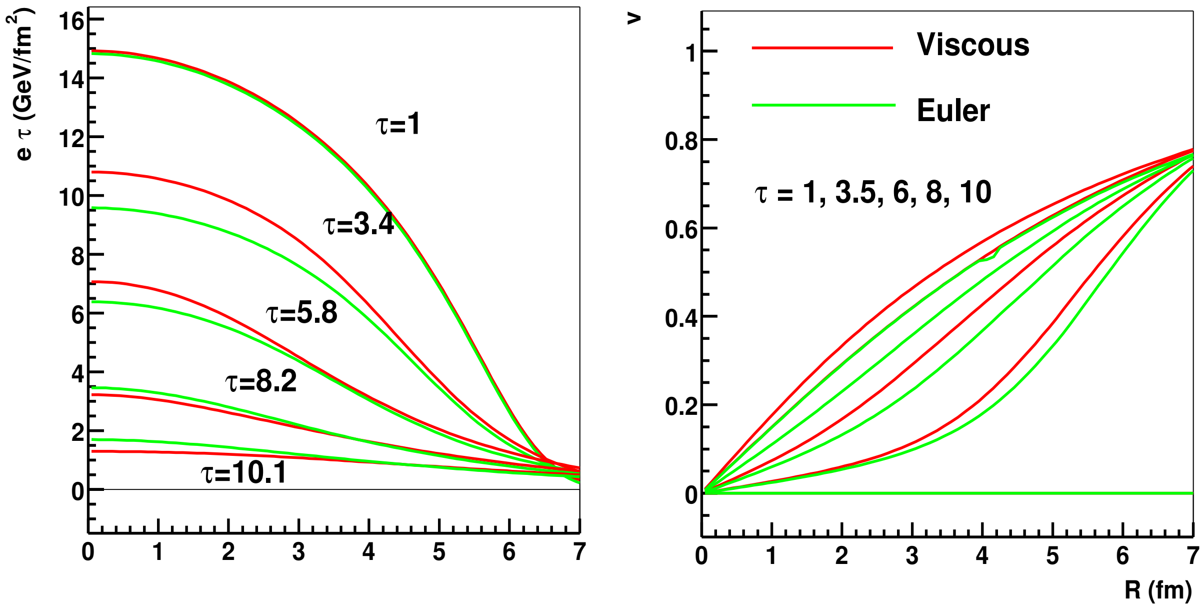
$$\frac{\Gamma_s}{\tau} \approx 0.2$$

- The models disagree with each other.
- T^{ij} is not asymptotic with $\sim \eta(\partial^i v^j + \partial j v^i - \frac{2}{3}\delta^{ij}\partial_l v^l)$

I can solve the Navier Stokes Equations

Bjorken Solution with transverse Expansion:

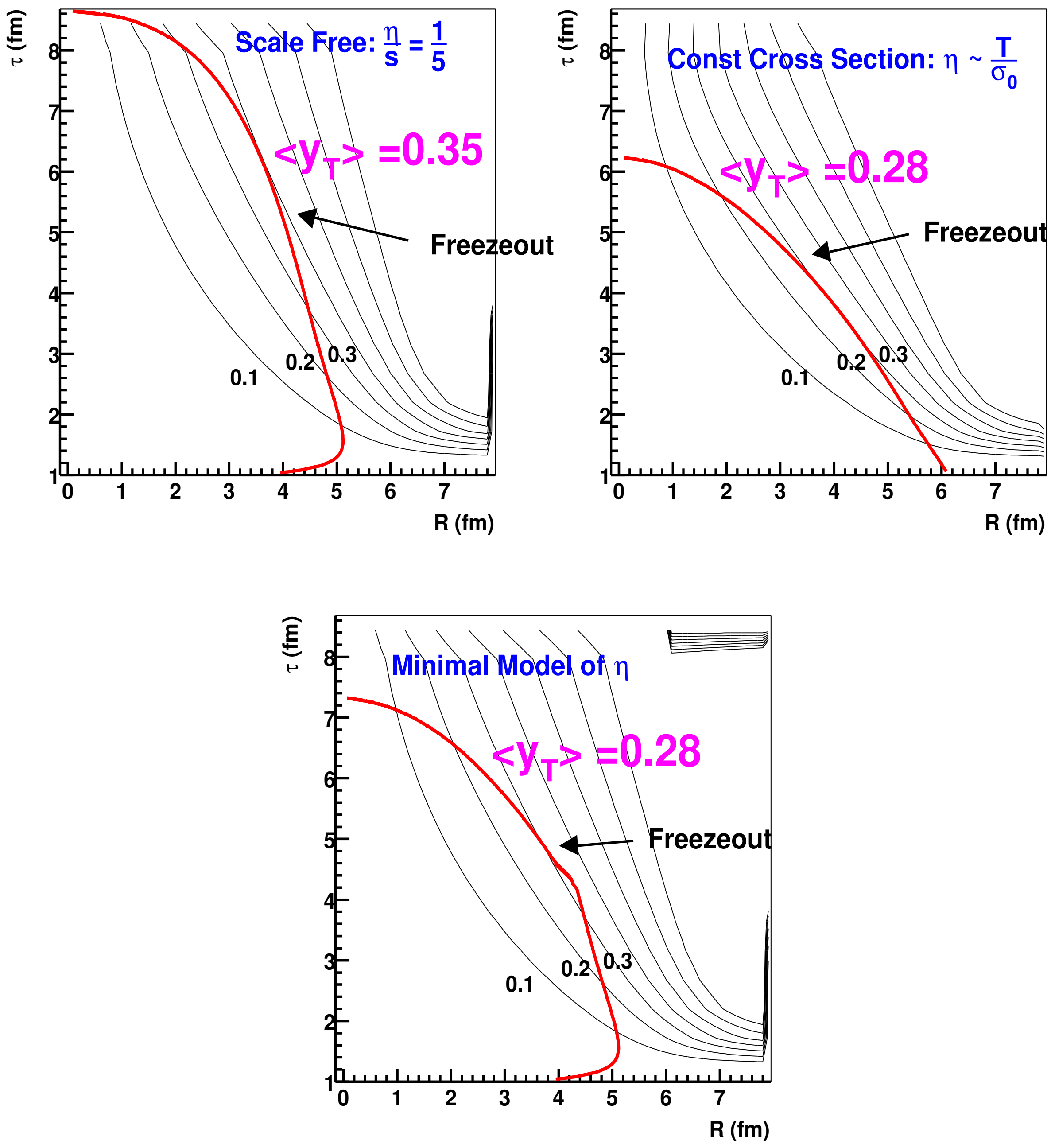
Initial conditions from Kolb's Hydro: $p = \frac{1}{3}e$ and $\eta/s = 0.2$



- First the viscous case does less longitudinal work.
- But, the viscous transverse velocity grows more rapidly because the transverse pressure is larger
- The larger transverse velocity then reduces the viscous energy density more quickly than Euler hydro.

Viscous corrections do NOT integrate to give an O(1) change to the flow.

Compare the three models of viscosity:



The minimal model of η and the Const X.-section model of η yield the same amount of radial flow.

Conclusions:

- Viscosity does not change the ideal hydrodynamic solution particularly much.
- Having a viscosity which is proportional to $\eta \sim T^3$ with a physically reasonable but small value of η/s reproduces the radial flow found by Denes with large cross sections and unphysical values of η/s.
- TO DO: Elliptic Flow