Fit a circle in 3D

Following method of Craig M. Shakarji of NIST

http://nvlpubs.nist.gov/nistpubs/jres/103/6/j36sha.pdf

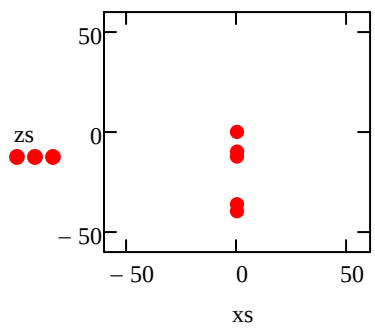
Steps:
1. Transform data points so that the centroid is at the origin
2. Fit a plane to the points
3. Rotate the date to that plane
4. Do a 2D circle fit to the data
5. Using the plane direction vector and the 2D circle values as the initial estimate of the fit parameters do a full 3D circle simultaneous fit with the 7 parameters (plane vector, circle center vector and radius). This fit is done in the centroid frame.
6. Transform back from the centroid frame to the lab frame.

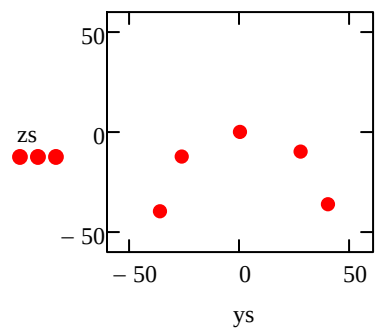
data input, turn off when using simulated data

$XG = \mathrm{READPRN}(\text{"CMMG2.txt"})$ units in mm

$X = XG$

$xs = X^{\langle 0 \rangle}$ $ys = X^{\langle 1 \rangle}$ $zs = X^{\langle 2 \rangle}$





each row is a point in x,y,z

$$X = \begin{pmatrix} -0.01852 & -0.03624 & 0.00365 \\ -0.00957 & -36.14912 & -39.67986 \\ -0.00299 & -26.29254 & -12.35234 \\ 0.00148 & 27.32683 & -9.85594 \\ -0.00203 & 39.67964 & -36.14912 \end{pmatrix}$$

data input, turn off when using simulated data

Simulation data for testing, turn off when using real data

Do the math

Fit points to a plane

$Xf = X$

Translate collection of points so that origin lies on the centroid

$$FC(X) = \left| \begin{array}{l} N \leftarrow \mathrm{last}\left(X^{\langle 0 \rangle}\right) + 1 \\ xm_0 \leftarrow \dfrac{\sum X^{\langle 0 \rangle}}{N} \\ xm_1 \leftarrow \dfrac{\sum X^{\langle 1 \rangle}}{N} \\ xm_2 \leftarrow \dfrac{\sum X^{\langle 2 \rangle}}{N} \\ xm \end{array} \right.$$

Finds the centroid

$dc = FC(Xf)$

$$TL(X, d) = \mathrm{augment}\left(X^{\langle 0 \rangle} - d_0, X^{\langle 1 \rangle} - d_1, X^{\langle 2 \rangle} - d_2\right)$$

linear translation

$Xf = TL(Xf, dc)$ collection now has orgin as the centroid

According to Sakarji the direction vector for the least sq. plane fit to the data is the eigen vector solution to the following equation with the smallest eigenvalue

$Xf^T \cdot Xf \cdot v = \lambda \cdot v$ where v are eigen vectors and λ are eigen values.

Xf^T*Xf is a 3 by 3 matrix

The sqrt of the eigenvalues can be obtained from Singular Value Decomposition (SVD)

$sol = svd2(Xf)$ mathcad SVD function

$$sol_0 = \begin{pmatrix} 65.740924 \\ 34.672308 \\ 0.013754 \end{pmatrix}$$

singular values

The objective function (Skarji again) is equal to the λs and we want the smallest one so

$$\lambda = \min\left(sol_0\right)^2$$

$$nv = eigenvec\left(Xf^T \cdot Xf, \lambda\right)$$

Using the mathcad function to find the eigenvector for this eigenvalue gives the direction vector of the plane. Sakarji may have found this eigen vector using SVD, but this appears to works too.

$$nv = \begin{pmatrix} 1 \\ -0.0001 \\ 0.000132 \end{pmatrix}$$

Rewriting all this as a single function

$$np(X) = \left| \begin{array}{l} sol \leftarrow svd2(X) \\ \lambda \leftarrow \min\left(sol_0\right)^2 \\ n \leftarrow eigenvec\left(X^T \cdot X, \lambda\right) \\ n \end{array} \right.$$

direction vector of the plane least sq fit to the points

$$A1 = np(Xf) = \begin{pmatrix} 1 \\ -0.0001 \\ 0.000132 \end{pmatrix}$$

Rotate collection of points so to the fit plane so that we can deal with the circle in 2D.

We define such that the data is in y z on the fit plane.

Put together the rotation matrix for this operation following the link below

$$tp(u) = \begin{bmatrix} \left(u_0\right)^2 & u_0 \cdot u_1 & u_0 \cdot u_2 \\ u_0 \cdot u_1 & \left(u_1\right)^2 & u_1 \cdot u_2 \\ u_0 \cdot u_2 & u_1 \cdot u_2 & \left(u_2\right)^2 \end{bmatrix} \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$cpm(u) = \begin{pmatrix} 0 & -u_2 & u_1 \\ u_2 & 0 & -u_0 \\ -u_1 & u_0 & 0 \end{pmatrix}$$

$$Rn\theta(n,\theta) = I\cdot\cos(\theta) + \sin(\theta)\cdot cpm(n) + (1-\cos(\theta))\cdot tp(n)$$

Rotation matrix link

$nx = stack(1,0,0)$ unit vector in x direction

$nraxis = A1 \times nx$ direction vector of plane crossed into x unit vector

$na = \dfrac{nraxis}{|nraxis|}$ unit vector about which to rotate

$\theta rot = acos(nx\cdot A1)$ angle to rotate

$\theta rot\cdot\dfrac{180}{\pi} = 0.009486$

$Xs = Xf$ save the centroid fram data for the full 3D fit to follow

$$Xf = \left(Rn\theta(na,\theta rot)\cdot Xf^{T}\right)^{T}$$

2D circle fit with method lifted from MathCad forum

Linear method to get initial guess

$u = 0\cdot Xf^{\langle 0\rangle} + 1$ vector of 1s

$$z2 = \overrightarrow{\left[\left(Xf^{\langle 1\rangle}\right)^{2} + Xf^{\langle 2\rangle}\right]}$$

$X2^{\langle 0\rangle} = Xf^{\langle 1\rangle}$ $X2^{\langle 1\rangle} = Xf^{\langle 2\rangle}$ $X2^{\langle 2\rangle} = u$

$$a2 = \left(X2^{T}\cdot X2\right)^{-1}\cdot\left(X2^{T}\cdot z2\right)$$

$$\begin{pmatrix} cx \\ cy \end{pmatrix} = \begin{pmatrix} a2_0 \\ a2_1 \end{pmatrix}\cdot\frac{1}{2}$$

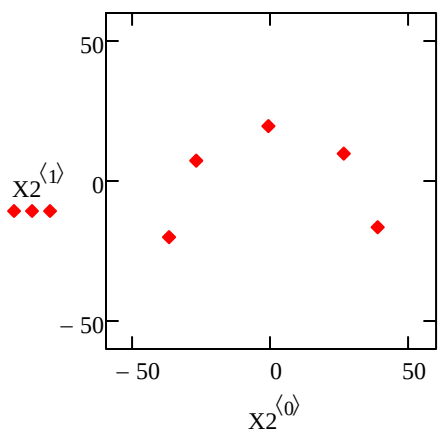
$$r = \sqrt{cx^{2} + cy^{2} + a2_2}$$

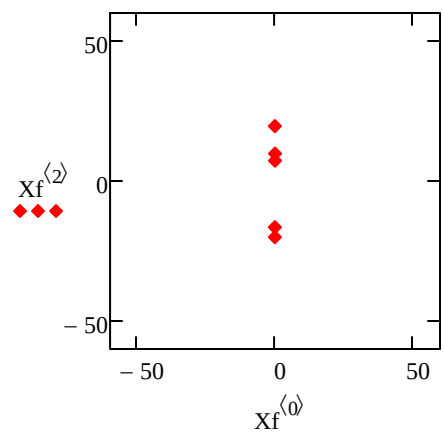
$$\begin{pmatrix} cx \\ cy \\ r \end{pmatrix} = \begin{pmatrix} 1.226134 \\ -16.290682 \\ 33.614385 \end{pmatrix}$$

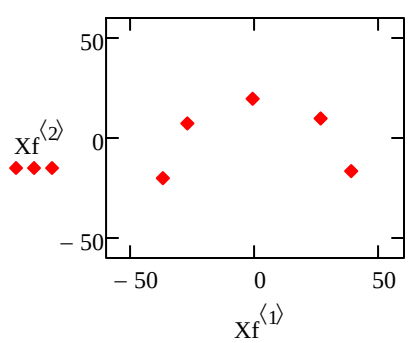
$$\mathrm{err}(cx, cy, r, X) = \left| \begin{array}{l} \text{for } i \in 0 .. \mathrm{rows}\left(X^{\langle 0 \rangle}\right) - 1 \\ \quad \left| \begin{array}{l} d \leftarrow \sqrt{\left[\left(X^{\langle 0 \rangle}\right)_i - cx\right]^2 + \left[\left(X^{\langle 1 \rangle}\right)_i - cy\right]^2} \\ err_i \leftarrow d - r \end{array} \right. \\ err \end{array} \right.$$

Given $\quad \mathrm{err}(cx, cy, r, X2) = 0 \quad \begin{pmatrix} cx \\ cy \\ r \end{pmatrix} = \mathrm{Minerr}(cx, cy, r) \quad \begin{pmatrix} cx \\ cy \\ r \end{pmatrix} = \begin{pmatrix} 0.859409 \\ -18.304013 \\ 37.955014 \end{pmatrix}$

$$\sigma est = \frac{\left|\mathrm{err}(cx, cy, r, X2)\right|}{\sqrt{\mathrm{last}\left(X2^{\langle 0 \rangle}\right) + 1 - 3}} = 0.002188$$







Do a complete simutaneous 3D refit using all the fit

parameters

This again is lifted from the work of [Craig M. Shakarji of NIST]

Use the parameters obtained above from the plane fit and the 2D circle fit above as a starting point.

$$g(xi, x, A) = \frac{A}{|A|} \cdot (xi - x)$$

distance from point xi to a plane defined by point x and direction vector A

$$f(xi, x, A) = \left| \frac{A}{|A|} \times (xi - x) \right|$$

distance between point xi and a line defined by A, direction vector and x, a point.

$$d(xi, x, A, r) = \sqrt{g(xi, x, A)^2 + (f(xi, x, A) - r)^2}$$

combined error or distance from the circle in both the out of plane direction and the out of radius direction

$$ds(xx, xy, xz, A, B, C, r, X) = \left| \begin{array}{l} Av \leftarrow (A \;\; B \;\; C)^T \\ xv \leftarrow (xx \;\; xy \;\; xz)^T \\ \text{for } i \in 0 .. \text{last}\left(X^{\langle 0 \rangle}\right) \\ \quad err_i \leftarrow d\left[\left(X^T\right)^{\langle i \rangle}, xv, Av, r\right] \\ err \end{array} \right.$$

vector of distances from the circle, one element for each data point.

This is what gets minimized.

$$dn(xx, xy, xz, A, B, C, X) = \left| \begin{array}{l} Av \leftarrow (A \;\; B \;\; C)^T \\ xv \leftarrow (xx \;\; xy \;\; xz)^T \\ \text{for } i \in 0 .. \text{last}\left(X^{\langle 0 \rangle}\right) \\ \quad err_i \leftarrow g\left[\left(X^T\right)^{\langle i \rangle}, xv, Av\right] \\ err \end{array} \right.$$

vector of distances from the plane, one element for each data point.

Will be used to show the errors

$$dr(xx, xy, xz, A, B, C, r, X) = \left| \begin{array}{l} Av \leftarrow (A \;\; B \;\; C)^T \\ xv \leftarrow (xx \;\; xy \;\; xz)^T \\ \text{for } i \in 0 .. last\left(X^{\langle 0 \rangle}\right) \\ \quad err_i \leftarrow f\left[\left(X^T\right)^{\langle i \rangle}, xv, Av\right] - r \\ err \end{array} \right.$$

vector of distances from the radius, one element for each data point.

Will be used to show the errors

$$xc = 0 \qquad Aa = A1_0$$
$$yc = cx \qquad Ab = A1_1$$
$$zc = cy \qquad Ac = A1_2$$

$$\begin{pmatrix} xc \\ yc \\ zc \\ Aa \\ Ab \\ Ac \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0.859409 \\ -18.304013 \\ 1 \\ -0.0001 \\ 0.000132 \\ 37.955014 \end{pmatrix}$$

do the minimization

Given $\quad ds(xc, yc, zc, Aa, Ab, Ac, r, Xs) = 0$

Xs is the collection of data points transformed to the centroid frame

$$\begin{pmatrix} xc \\ yc \\ zc \\ Aa \\ Ab \\ Ac \\ r \end{pmatrix} = Minerr(xc, yc, zc, Aa, Ab, Ac, r)$$

Minimize ds using Levenberg - Marguardt to find 7 parameters specifying circle in 3 space.

Result after fitting the circle to the data

$$\begin{pmatrix} xc \\ yc \\ zc \\ Aa \\ Ab \\ Ac \\ r \end{pmatrix} = \begin{pmatrix} 0.002496 \\ 0.859409 \\ -18.304013 \\ 1.008038 \\ -0.000101 \\ 0.000133 \\ 37.955014 \end{pmatrix} \qquad uf = \text{submatrix}\left[\begin{pmatrix} xc \\ yc \\ zc \\ Aa \\ Ab \\ Ac \\ r \end{pmatrix}, 3, 5, 0, 0\right] \qquad cf = \text{submatrix}\left[\begin{pmatrix} xc \\ yc \\ zc \\ Aa \\ Ab \\ Ac \\ r \end{pmatrix}, 0, 2, 0, 0\right]$$

circle center in the centroid frame

$$uf = \frac{uf}{|uf|}$$

unit vector normal to the circle plane

Generate circle of points using fit parameters

$$Nc = 200 \qquad i = 0..Nc \qquad \Delta\theta = \frac{2\cdot\pi}{Nc}$$

$$\theta c_i = i\cdot\Delta\theta \qquad xf = \overrightarrow{(0\cdot\theta c)} \qquad yf = \overrightarrow{(r\cdot\cos(\theta c))} \qquad zf = \overrightarrow{(r\cdot\sin(\theta c))}$$

$$Xcf = \text{augment}(xf, yf, zf)$$

$$\text{cos_u_to_x} = uf\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \theta sw = \text{acos}(\text{cos_u_to_x})$$

angle between circle plane vector and the y axis

$$nra = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times uf = \begin{pmatrix} 0 \\ -0.000132 \\ -0.0001 \end{pmatrix} \qquad nra = \frac{nra}{|nra|}$$

unit vector for axis about which will rotate

$$Xcfr = \left(Rn\theta(nra, \theta sw)\cdot Xcf^T\right)^T$$

Rotating the generated points into the circle plane

$$Xcfr = TL\left[Xcfr, -\begin{pmatrix} xc \\ yc \\ zc \end{pmatrix}\right]$$

move the generated points to the determened place in the centroid frame

$$Xcfr = TL(Xcfr, -dc)$$

Transform the circle generated points back to the orginal data starting frame from the centroid frame

$$c = TL\left(cf^T, -dc\right)^T$$

Transform the fit center from the centroid frame to the starting frame

$$\mathrm{sprd}(v) = \max(v) - \min(v)$$

$$p(c, X) = \left| \begin{array}{l} pa \leftarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \text{for } i \in 0 .. \mathrm{last}\left(X^{\langle 0 \rangle}\right) \\ \quad \left| \begin{array}{l} p_i \leftarrow \left(X^T\right)^{\langle i \rangle} - c \\ pa \leftarrow \mathrm{augment}\left(pa, p_i\right) \end{array} \right. \\ pa \leftarrow \mathrm{submatrix}\left(pa, 0, 2, 1, \mathrm{last}\left(X^{\langle 0 \rangle}\right) + 1\right) \\ pa \leftarrow pa^T \end{array} \right.$$

point vectors from the fit circle center

$$\mathrm{ang2}(c, X) = \left| \begin{array}{l} pp \leftarrow p(c, X) \\ \text{for } i \in 0 .. \mathrm{last}\left(pp^{\langle 0 \rangle}\right) \\ \quad \left| \begin{array}{l} ca_i \leftarrow \dfrac{\left(pp^T\right)^{\langle 0 \rangle} \cdot \left(pp^T\right)^{\langle i \rangle}}{\left|\left(pp^T\right)^{\langle 0 \rangle}\right| \cdot \left|\left(pp^T\right)^{\langle i \rangle}\right|} \\ ang_i \leftarrow \mathrm{acos}\left(ca_i\right) \end{array} \right. \\ ang \cdot \dfrac{180}{\pi} \end{array} \right.$$

angle of the points about the circle center relative to one of the points in degrees

Do the math

Results

units: mm

$$c = \begin{pmatrix} -0.00383 \\ 1.765123 \\ -37.910735 \end{pmatrix}$$

circle center

$$A1 = \begin{pmatrix} 1 \\ -0.0001 \\ 0.000132 \end{pmatrix}$$

circle normal

$$r = 37.955014$$

circle radius

$$drv = dr(xc, yc, zc, Aa, Ab, Ac, r, Xs)$$

radial distance of points from r

$$drv = \begin{pmatrix} 0.00214 \\ 0.000481 \\ -0.00157 \\ -0.001459 \\ 0.000405 \end{pmatrix}$$

$$Stdev(drv) = 0.001547$$

standard deviation of the points from r

$$Stdev(drv) \cdot \sqrt{12} = 0.005358$$

window spread

$$\frac{Stdev(drv) \cdot \sqrt{12}}{2} = 0.002679$$

or +- value

$$sprd(drv) = 0.00371$$

this sample spread

$$dnv = dn(xc, yc, zc, Aa, Ab, Ac, Xs)$$

distance of points from the circle plane

$$dnv = \begin{pmatrix} -0.009517 \\ -0.00217 \\ 0.007019 \\ 0.00644 \\ -0.00177 \end{pmatrix}$$

$$Stdev(dnv) = 0.006877$$

standard deviation distance from the plane

$$Stdev(dnv) \cdot \sqrt{12} = 0.023822$$

window spread of points from the plane

$$sprd(dnv) = 0.016536$$

this sample spread in distance from the plane

$$dwn = \frac{dnv}{r} = \begin{pmatrix} -0.000251 \\ -0.000057 \\ 0.000185 \\ 0.00017 \\ -0.000047 \end{pmatrix} \cdot rad$$

wobble of the rotation axis implied by the distance of the points from the circle plane

$$Stdev(dwn) = 0.000181 \cdot rad$$

standard deviation of wobble

$$Stdev(dwn) \cdot \sqrt{12} = 0.000628 \cdot rad$$

window spread

$$Stdev(dwn) \cdot \sqrt{12} = 129.461961 \cdot arcsec$$

there is no spec given for axis wobble on the MTR rotary head, but the rotation angle spec is +- 5 arc sec which is 13 times less than this wobble.

$L = 300$ length of the sector

$$dwn \cdot L = \begin{pmatrix} -0.075227 \\ -0.017154 \\ 0.055479 \\ 0.050904 \\ -0.013993 \end{pmatrix}$$

projected error at the end of the sector due to axis wobble

$$Stdev(dwn \cdot L) \cdot \sqrt{12} = 0.188295$$

window spread in the projected error at the end of the sector due to axis wobble

$$p(c, X) = \begin{pmatrix} -0.01469 & -1.801363 & 37.914385 \\ -0.00574 & -37.914243 & -1.769125 \\ 0.00084 & -28.057663 & 25.558395 \\ 0.00531 & 25.561707 & 28.054795 \\ 0.0018 & 37.914517 & 1.761615 \end{pmatrix}$$

Measured position vectors relative to the fit center of the circle

$$\text{angmeasured} = \text{ang2}(c, X) = \begin{pmatrix} 0 \\ 89.951397 \\ 44.948715 \\ 45.057912 \\ 90.059945 \end{pmatrix}$$

angle between the first point vector and all the point vectors
in degrees

$$\text{angset} = \begin{pmatrix} 0 \\ 90 \\ 45 \\ 45 \\ 90 \end{pmatrix}$$

values set for rotary head

$$\delta\theta = \text{angmeasured} - \text{angset} = \begin{pmatrix} 0 \\ -0.048603 \\ -0.051285 \\ 0.057912 \\ 0.059945 \end{pmatrix}$$

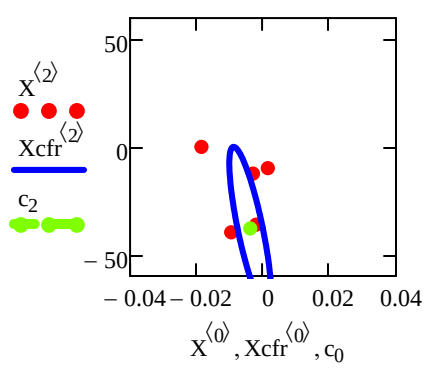
discrepancy between measured and set values in degrees

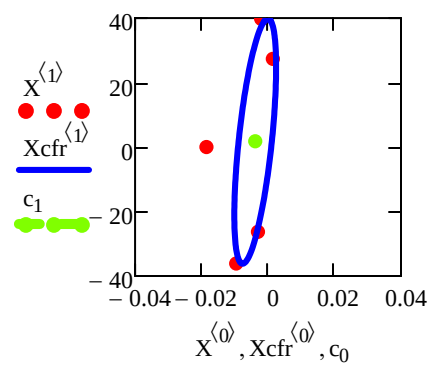
$$\theta\text{spread} = \text{sprd}(\delta\theta)\cdot\frac{\pi}{180} = 400.428427\cdot\text{arcsec}$$

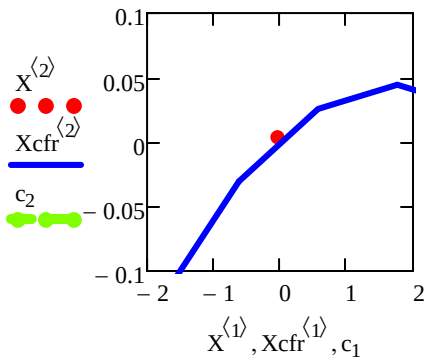
This is 40 times the +- 5 arcsec spec given for the MTR rotary head

units is mm

plotting x limits $\quad \text{xmin} = -.04 \quad \text{xmax} = .04$






0.1
0.05
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X⟨1⟩, Xcfr⟨1⟩, c1

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