Evolution of Generalized Parton Distributions

Marc Kirch

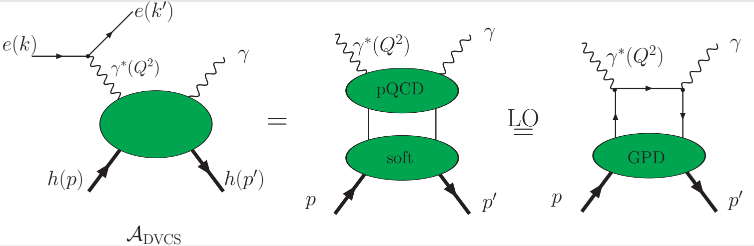
Institute for theoretical Physics II
Ruhr-University Bochum

Workshop on Future Prospects of QCD at High Energies
Brookhaven National Laboratory, USA
17. - 22.07.2006

Outline

Deeply Virtual Compton Scattering

Standard Example: Deeply Virtual Compton Scattering $ep \longrightarrow ep\gamma$



Exclusive Process: Factorization of the **Amplitude**

$$\mathcal{A}_{\mathrm{DVCS}} \propto C_{\mathrm{pQCD}} * \boldsymbol{F}_{\mathrm{soft}}$$

F_{soft}: **universal** generalized parton distribution (GPD)

Generalized Parton Distributions (GPDs)

E.g.: Unpolarized Quark GPD

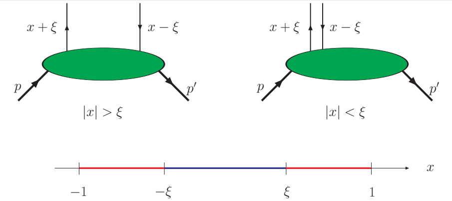
$$F(x,\xi,t) = \int \frac{\mathrm{d}z}{2\pi} e^{ixz} \langle p'|\overline{q}(-\frac{z}{2})\gamma^+ q(\frac{z}{2})|p\rangle$$

bilocal Operators

Bilocal Quark- (Gluon) Operator

$$\mathcal{O}(z_1,z_2) = \overline{q}(z_1)\gamma^+ q(z_2)$$

Range of kin. Variables



kinematical Variables

$$x, \quad \xi = \frac{p^+ - p'^+}{p^+ + p'^+}$$

$$\Delta = p' - p, \quad t = \Delta^2$$

Generalized Parton Distributions (GPDs)

E.g.: Unpolarized Quark GPD

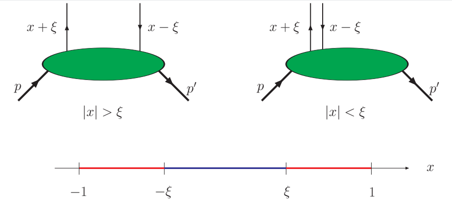
$$F(x,\xi,t) = \int \frac{dz}{2\pi} e^{ixz} \langle \mathbf{p'} | \bar{q}(-\frac{z}{2})\gamma^+ q(\frac{z}{2}) | \mathbf{p} \rangle$$

bilocal Operators

Bilocal Quark- (Gluon) Operator

$$\mathcal{O}(z_1,z_2) = \bar{q}(z_1)\gamma^+ q(z_2)$$

Range of kin. Variables



kinematical Variables

$$x, \quad \xi = \frac{p^+ - p'^+}{p^+ + p'^+}$$

$$\Delta = p' - p, \quad t = \Delta^2$$

Generalized Parton Distributions (GPDs)

E.g.: Unpolarized Quark GPD

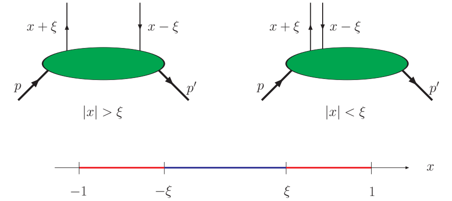
$$F(x,\xi,t) = \int \frac{dz}{2\pi} e^{ixz} \langle p' | \bar{q}(-\frac{z}{2}) \gamma^+ q(\frac{z}{2}) | p \rangle$$

bilocal Operators

Bilocal Quark- (Gluon) Operator

$$\mathcal{O}(z_1, z_2) = \bar{q}(z_1) \gamma^+ q(z_2)$$

Range of kin. Variables



kinematical Variables

$$x, \quad \xi = \frac{p^+ - p'^+}{p^+ + p'^+}$$

$$\Delta = p' - p, \quad t = \Delta^2$$

Generalized Parton Distributions (GPDs)

E.g.: Unpolarized Quark GPD

$$F(x,\xi,t) = \int \frac{dz}{2\pi} e^{ixz} \langle p' | \overline{q}(-\frac{z}{2}) \gamma^+ q(\frac{z}{2}) | p \rangle$$

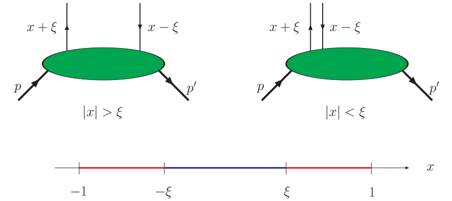
$$= \boldsymbol{H(x,\xi,t)} \overline{u}(p') \gamma^+ u(p) + \boldsymbol{E(x,\xi,t)} \overline{u}(p') \frac{i\sigma^{+\mu}\Delta_\mu}{2m} u(p)$$

bilocal Operators

Bilocal Quark- (Gluon) Operator

$$\mathcal{O}(z_1,z_2) = \overline{q}(z_1)\gamma^+ q(z_2)$$

Range of kin. Variables



kinematical Variables

$$x, \quad \xi = \frac{p^+ - p'^+}{p^+ + p'^+}$$

$$\Delta = p' - p, \quad t = \Delta^2$$

Generalized Parton Distributions (GPDs)

E.g.: Unpolarized Quark GPD

$$F(x,\xi,t) = \int \frac{dz}{2\pi} e^{ixz} \langle p' | \overline{q}(-\frac{z}{2})\gamma^+ q(\frac{z}{2}) | p \rangle$$

$$= \int \frac{dz}{2\pi} e^{ixz} \boldsymbol{f(-\frac{z}{2}, \frac{z}{2})}$$

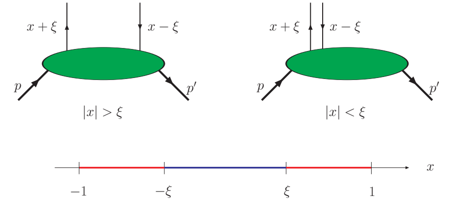
bilocal Operators

Bilocal Quark- (Gluon) Operator

$$\mathcal{O}(z_1, z_2) = \overline{q}(z_1)\gamma^+ q(z_2)$$

$$\boldsymbol{f(z_1, z_2) = \langle p' | \mathcal{O}(z_1, z_2) | p \rangle}$$

Range of kin. Variables



kinematical Variables

$$x, \quad \xi = \frac{p^+ - p'^+}{p^+ + p'^+}$$

$$\Delta = p' - p, \quad t = \Delta^2$$

Properties of GPDs

- GPDs **interpolate** between
  - (forward) Parton Distributions ($\xi \longrightarrow 0$)
  - Formfators (x-Moments)
- kinematical Region $|x| < \xi$ Source of **new Information**!
  $q\overline{q}$-Configurations in Hadrons!
- Ji Sumrule: Contribution of Quark-Angular Momentum to the Nucleon spin.

$$J^q = \frac{1}{2} \int \mathrm{d}x\, x (H(x,\xi) + E(x,\xi))$$

[Ji, *Phys. Rev. Lett.* 78, (97)]

- Multidimensional Spatial picture of Hadron structure (Femtoscopy)

[M. Burkardt, *Int. J. Mod. Phys*, A18, (03)],

[M. Diehl, *Eur. Phys. J.*, C25, (02)]

Experimental Access

Measurement of DVCS at e.g.

- CLAS@CEBAF/JLAB (Spin-Asymmetries)
- HERMES@HERA/DESY: (Charge- & Spin- Asymmetries)
- Experiment: Energy-Scale $\mathbf{Q^2}$

And: Scale dependence!

- The GPD depends on scale Q^2

$$F = F(x, \xi; \mathbf{Q^2})$$

Evolution

Scale Dependence

- The GPD is a function of Q^2

Momentum Space

$$F = F(x, \xi; \mathbf{Q^2})$$

- Physical: “Resolution”

Evolution

Scale Dependence

- The GPD is a function of Q^2

Coordinate Space

$$f = f(z_1, z_2; \boldsymbol{Q^2})$$

- Physical: “Resolution”
- Technical: Renormalization of composite operators

Evolution

Scale Dependence

- The GPD is a function of Q^2

Coordinate Space

$$f = f(z_1, z_2; \mathbf{Q^2})$$

- Physical: "Resolution"
- Technical: Renormalization of composite operators

Evolution Equation

- Scale dependence governed by

$$Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} f(z_1, z_2; Q^2) = [\mathbb{H} f]\,(z_1, z_2; Q^2)$$

- Evolution kernel:

$$\mathbb{H} = \boldsymbol{\alpha_s} \mathbb{H}_\mathbf{1} + \alpha_s^2 \mathbb{H}_2 + \cdots$$

[Balitsky, Braun, *Nucl. Phys.*, B311, (89)]

Evolution

Scale Dependence

- The GPD is a function of Q^2

Coordinate Space

$$f = f(z_1, z_2; \boldsymbol{Q^2})$$

- Physical: “Resolution”
- Technical: Renormalization of composite operators

Solution

- How does GPD change with change of scale $Q_1^2 \to Q_2^2$?

$$f(z_1, z_2; \boldsymbol{Q_1^2}) \longrightarrow f(z_1, z_2; \boldsymbol{Q_2^2})$$

Evolution Equation

- Scale dependence governed by

$$Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} f(z_1, z_2; Q^2) = [\mathbb{H} f](z_1, z_2; Q^2)$$

- Evolution kernel:

$$\mathbb{H} = \boldsymbol{\alpha_s \mathbb{H}_1} + \alpha_s^2 \mathbb{H}_2 + \cdots$$

[Balitsky, Braun, *Nucl. Phys.*, B311, (89)]

Evolution

Scale Dependence

- The GPD is a function of Q^2

Coordinate Space

$$f = f(z_1, z_2; \mathbf{Q^2})$$

- Physical: “Resolution”
- Technical: Renormalization of composite operators

Evolution Equation

- Scale dependence governed by

$$Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} f(z_1, z_2; Q^2) = [\mathbb{H} f]\,(z_1, z_2; Q^2)$$

- Evolution kernel:

$$\mathbb{H} = \boldsymbol{\alpha_s \mathbb{H}_1} + \alpha_s^2 \mathbb{H}_2 + \cdots$$

[Balitsky, Braun, *Nucl. Phys.*, B311, (89)]

Solution

- How does GPD change with change of scale $Q_1^2 \to Q_2^2$?

$$F(x, \xi; \mathbf{Q_1^2}) \longrightarrow F(x, \xi; \mathbf{Q_2^2})$$

FT: In **momentum** space

Evolution

Scale Dependence

- The GPD is a function of Q^2

Coordinate Space

$$f = f(z_1, z_2; \mathbf{Q^2})$$

- Physical: "Resolution"
- Technical: Renormalization of composite operators

Evolution Equation

- Scale dependence governed by

$$Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} f(z_1, z_2; Q^2) = [\mathbb{H} f](z_1, z_2; Q^2)$$

- Evolution kernel:

$$\mathbb{H} = \boldsymbol{\alpha_s} \mathbb{H}_\mathbf{1} + \alpha_s^2 \mathbb{H}_2 + \cdots$$

[Balitsky, Braun, *Nucl. Phys.*, B311, (89)]

Solution

- How does GPD change with change of scale $Q_1^2 \to Q_2^2$?

$$F(x, \xi; \mathbf{Q_1^2}) \longrightarrow F(x, \xi; \mathbf{Q_2^2})$$

FT: In **momentum** space

Why important?

- Experiment & Theory: intrinsic **different scales** Q^2
- Model-independent QCD prediction.

How to Solve the Evolution Equation I

$$Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} f(z_1, z_2; Q^2) = -\alpha_s \left[\mathbb{H} f\right](z_1, z_2; Q^2)$$

- Strategy: Expand GPD $f(z_1, z_2)$ in set of eigenfunctions $\{f_j(z_1, z_2)\}$ of $\mathbb{H}$
- If $f_j(z_1, z_2)$ is **eigenfunction** of $\mathbb{H}$

$$Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} f_j(z_1, z_2; Q^2) = -\boldsymbol{\alpha_s(Q^2)}\, \mathbb{H} f_j(z_1, z_2; Q^2)$$

How to Solve the Evolution Equation I

$$Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} f(z_1, z_2; Q^2) = -\alpha_s \left[\mathbb{H} f\right](z_1, z_2; Q^2)$$

- Strategy: Expand GPD $f(z_1, z_2)$ in set of eigenfunctions $\{f_j(z_1, z_2)\}$ of $\mathbb{H}$
- If $f_j(z_1, z_2)$ is **eigenfunction** of $\mathbb{H}$

$$Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} f_j(z_1, z_2; Q^2) = -\boldsymbol{\alpha_s(Q^2)\,\gamma(j)}\, f_j(z_1, z_2; Q^2)$$

How to Solve the Evolution Equation I

$$Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} f(z_1, z_2; Q^2) = -\alpha_s \left[\mathbb{H} f\right](z_1, z_2; Q^2)$$

- Strategy: Expand GPD $f(z_1, z_2)$ in set of eigenfunctions $\{f_j(z_1, z_2)\}$ of $\mathbb{H}$
- If $f_j(z_1, z_2)$ is **eigenfunction** of $\mathbb{H}$

$$Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} f_j(z_1, z_2; Q^2) = -\boldsymbol{\alpha_s(Q^2)\,\gamma(j)}\, f_j(z_1, z_2; Q^2)$$

- can be easiliy integrated

$$f_j(z_1, z_2; \boldsymbol{Q_2^2}) = \left(\frac{\boldsymbol{\alpha_s(Q_1^2)}}{\boldsymbol{\alpha_s(Q_2^2)}}\right)^{-\gamma(j)} f_j(z_1, z_2; \boldsymbol{Q_1^2})$$

How to Solve the Evolution Equation I

$$Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} f(z_1, z_2; Q^2) = -\alpha_s \left[\mathbb{H} f\right](z_1, z_2; Q^2)$$

- Strategy: Expand GPD $f(z_1, z_2)$ in set of eigenfunctions $\{f_j(z_1, z_2)\}$ of $\mathbb{H}$
- If $f_j(z_1, z_2)$ is **eigenfunction** of $\mathbb{H}$

$$Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} f_j(z_1, z_2; Q^2) = -\boldsymbol{\alpha_s(Q^2)\,\gamma(j)}\, f_j(z_1, z_2; Q^2)$$

- can be easiliy integrated

$$f_j(z_1, z_2; \boldsymbol{Q_2^2}) = \left(\frac{\alpha_s(Q_1^2)}{\alpha_s(Q_2^2)}\right)^{-\gamma(j)} f_j(z_1, z_2; \boldsymbol{Q_1^2})$$

- **If** $\{f_j(z_1, z_2)\}$ **complete orthonormal** set $\Rightarrow$ expansion coefficients simply obtained by projection!

$$f(z_1, z_2) = \sum_j c_j\, f_j(z_1, z_2), \qquad \boldsymbol{c_j = (f_j\,,\, f)}$$

How to solve the evolution equation II

- Eigenfunctions of $\mathbb{H}$ are known! (Bessel functions)

  [Efremov, Radyushkin, *Theor. Math. Phys*, 42 (80)]
- But **DO NOT** form an orthonormal set in whole range where GPDs are defined

Up to now:

- No closed **analytical** solution valid in the whole range where GPDs are defined.
- $\xi \to 1$: ERBL evolution (Meson wavefunction, Gegenbauer)
- $\xi \to 0$: DGLAP evolution (diagonal PDFs, Mellin transform)
- $0 < \xi < 1$: Regions $|x| < \xi$ (ERBL) and $|x| > \xi$ (DGLAP) have to be matched together. **Non-trivial!**

  [Belitsky et al., *Phys. Lett.* B421, (98)], [Kivel, Mankiewicz, *Nucl. Phys.* B557, (99)],

  [Shuvaev, *Phys. Rev.* D60, (99)]
- **Numerical** Vinnikov, hep-ph/0604284

Solution of the Evolution equation I

New solution

- Make **rigorously** use of (conformal) symmetry!

Conformal symmetry

- $\mathbb{H}$ is invariant under conformal transformations on the light cone: $\boldsymbol{SL(2,\mathbb{R})}$
- **Representation theory** of $SL(2,\mathbb{R})$ allows for construction of “suitable” Hilbert space in which GPD lives
- Hilbert space $\Rightarrow$ **Basis**-Decomposition of GPD ...
- ... (fortunately!) over Eigenfunktions of $\mathbb{H}$

[Manashov, M.K., Schäfer, *Phys. Rev. Lett.* 95, (05)], [M.K., Manashov, Schäfer, *Phys. Rev.* D72, (05)]

[Müller, Schäfer, *Nucl. Phys.*, B739 (06)]

Solution of the Evolution Equation II

- GPD: product space of unitary representations of $SL(2,\mathbb{R})$ group

$$f(z_1, z_2) \in V_1 \otimes V_2$$

- Representation theory: Decomposition over irreducible subspaces

$$V_1 \otimes V_2 = \bigoplus V_j \oplus \int \mathrm{d}^{\oplus} j \, T^j$$

- Allows expansion of GPD in **complete orthonormal** set (Basis) in coordinate space

$$f(z_1, z_2) = \sum c_j \, f_j(z_1, z_2) + \int \mathrm{d}j \, \tilde{c}_j \, \tilde{f}_j(z_1, z_2)$$

- $f_j(z_1, z_2)$ eigenfunctions, $\tilde{f}_j(z_1, z_2)$ can be rewritten in terms of f_j
- Fourier transform, Done

Solution in momentum space

Solution: Expansion over Basis

$$F(x,\xi,Q_2^2) = \sum_{j=3}^{\infty} p_j\left(\frac{x}{\xi}\right) L^{-\gamma(j)} c_\xi(j) + \int \frac{\mathrm{d}j}{i\pi} q_j\left(\frac{x}{\xi}\right) L^{-\gamma(j)} \tilde{c}_\xi(j)$$

- $c(j)$ - Expansion coefficients: Input GPD at $Q^2 = Q_1^2$

$$c_\xi(j) = \int \mathrm{d}x\, p_j\left(\frac{x}{\xi}\right) F(x,\xi,t;Q_1^2)$$

- $p_j(x)$, $q_j(x)$ - Basisvectors (Legendrefunctions)
- $L = \frac{\alpha_s(Q_1^2)}{\alpha_s(Q_2^2)}$
- $\gamma(j)$ anomalous dimension

Properties

- Completely analytical Form of solution: Well defined in whole range.
- Good convergence
- **Simple**!
- Small x and ξ: how big is Skewness-effect?
- Asymptotic form for $Q^2 \to \infty$
- Construction of numerical algorithm

Solution in momentum space

Solution: Expansion over Basis

$$F(x,\xi,Q_2^2) = \sum_{j=3}^{\infty} p_j\left(\frac{x}{\xi}\right) L^{-\gamma(j)} \mathbf{c}_\xi(\mathbf{j}) + \int \frac{\mathrm{d}j}{i\pi} q_j\left(\frac{x}{\xi}\right) L^{-\gamma(j)} \tilde{\mathbf{c}}_\xi(\mathbf{j})$$

- $c(j)$ - Expansion coefficients: Input GPD at $Q^2 = Q_1^2$

$$\mathbf{c}_\xi(\mathbf{j}) = \int \mathrm{d}x\, p_j\left(\frac{x}{\xi}\right) F(x,\xi;\mathbf{Q_1^2})$$

- $p_j(x)$, $q_j(x)$ - Basisvectors (Legendrefunctions)
- $L = \frac{\alpha_s(Q_1^2)}{\alpha_s(Q_2^2)}$
- $\gamma(j)$ anomalous dimension

Properties

- Completely analytical Form of solution: Well defined in whole range.
- Good convergence
- **Simple**!
- Small x and ξ: how big is Skewness-effect?
- Asymptotic form for $Q^2 \to \infty$
- Construction of numerical algorithm

Solution in momentum space

Solution: Expansion over Basis

$$F(x,\xi,Q_2^2) = \sum_{j=3}^{\infty} p_j\left(\frac{x}{\xi}\right) L^{-\gamma(j)} c_\xi(j) + \int \frac{\mathrm{d}j}{i\pi} q_j\left(\frac{x}{\xi}\right) L^{-\gamma(j)} \tilde{c}_\xi(j)$$

- $c(j)$ - Expansion coefficients: Input GPD at $Q^2 = Q_1^2$

$$c_\xi(j) = \int \mathrm{d}x\, p_j\left(\frac{x}{\xi}\right) F(x,\xi,t;Q_1^2)$$

- $p_j(x)$, $q_j(x)$ - Basisvectors (Legendrefunctions)
- $L = \frac{\alpha_S(Q_1^2)}{\alpha_S(Q_2^2)}$
- $\gamma(j)$ anomalous dimensions

Properties

- Completely analytical Form of solution: Well defined in whole range.
- Good convergence
- **Simple**!
- Small x and ξ: how big is Skewness-effect?
- Asymptotic form for $Q^2 \to \infty$
- Construction of numerical algorithm

Solution in momentum space

Solution: Expansion over Basis

$$F(x,\xi,\mathbf{Q_2^2}) = \sum_{j=3}^{\infty} p_j\left(\frac{x}{\xi}\right) \mathbf{L}^{-\gamma(j)} c_\xi(j) + \int \frac{\mathrm{d}j}{i\pi} q_j\left(\frac{x}{\xi}\right) \mathbf{L}^{-\gamma(j)} \tilde{c}_\xi(j)$$

- $c(j)$ - Expansion coefficients: Input GPD at $Q^2 = Q_1^2$

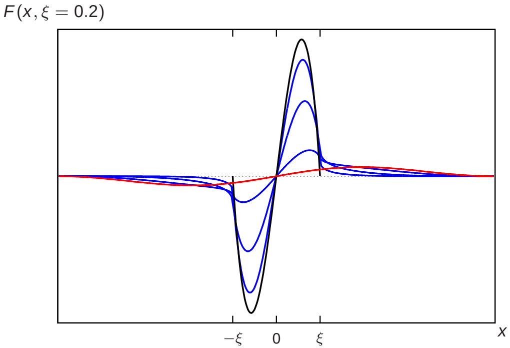
$$c_\xi(j) = \int \mathrm{d}x\, p_j\left(\frac{x}{\xi}\right) F(x,\xi,t;Q_1^2)$$

- $p_j(x)$, $q_j(x)$ - Basisvectors (Legendrefunctions)
- $L = \frac{\alpha_S(Q_1^2)}{\alpha_S(Q_2^2)}$
- $\gamma(j)$ anomalous dimensions

Properties

- Completely analytical Form of solution: Well defined in whole range.
- Good convergence
- **Simple**!
- Small x and ξ: how big is Skewness-effect?
- Asymptotic form for $Q^2 \to \infty$
- Construction of numerical algorithm

Numerics, Example



Evolution of an input GPD from scale $Q_1^2 = 1GeV^2$ (red)

to $Q_2^2 = 2GeV^2$, $10GeV^2$ und $100GeV^2$ (blue)

(slow) convergence to asymptotic form (black)

Summary

- For the first time: Complete analytical solution of the (LO) evolution equation for GPDs
- Valid in whole kinematical range where GPDs are defined
- Completely determined by conformal symmetry!
- Potential for various further analytical and numerical studies

Summary

- For the first time: Complete analytical solution of the (LO) evolution equation for GPDs
- Valid in whole kinematical range where GPDs are defined
- Completely determined by conformal symmetry!
- Potential for various further analytical and numerical studies

Thank you very much for your attention!